Name ______________________________ Date __________ Class __________

LESSON 7-6

Challenge

Exploring the Number e

John Napier, the inventor of logarithms in 1614, based his work on a number the Swiss mathematician Leonard Euler later called *e*. The value of *e* is the irrational number 2.71828 . . .

As you have seen, one way to approximate the value of *e* is to let the value of *n* become very large in the sequence of numbers obtained from the expression $1\left(1 + \frac{1}{n}\right)^n$. You can explore some other methods for evaluating *e*.

Consider the sequence $1, \frac{1}{1}, \frac{1}{2 \cdot 1}, \frac{1}{3 \cdot 2 \cdot 1}, \ldots$

1. Write the 9th term of the sequence. ______________________
2. Using a calculator, determine
   a. the sum of the first 5 terms of the sequence. ______________________
   b. the sum of the first 7 terms of the sequence. ______________________
   c. the sum of the first 10 terms of the sequence. ______________________
3. Use what you know about the value of *e* and the results of Exercise 2 to write an expression for *e* in terms of the given sequence of numbers. ______________________

A *continued fraction* is formed by a number added to a fraction whose denominator is a fraction added to a fraction whose denominator is a fraction, and so on, forming a pattern.

• **Example** To evaluate, start with the last denominator.

$$1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4}}} = 1 + \cfrac{1}{2 + \cfrac{1}{\frac{13}{4}}} = 1 + \cfrac{1}{2 + \frac{4}{13}} = 1 + \cfrac{1}{\frac{30}{13}} = 1 + \frac{13}{30} = \frac{43}{30}$$

Start.

Complete the continued fraction by finding the missing denominator. Then evaluate.

4. $2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{}}}}}$

5. $1 + \cfrac{2}{1 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cfrac{1}{}}}}}$

______________________ ______________________

6. Continue the pattern further in the fractions above and make an observation.

____

LESSON 7-6

Reteach

The Natural Base, e (continued)

The natural base, e, appears in the formula for interest compounded continuously.

$A = Pe^{rt}$

A = total amount

P = principal, or initial amount

r = annual interest rate

t = time in years

What is the total amount for an investment of $2000 invested at 3% and compounded continuously for 5 years?

Step 1 Identify the values that correspond to the variables in the formula.

P = initial investment = $2000

$r = 3\% = 0.03$

$t = 5$

Step 2 Substitute the known values into the formula.

$A = Pe^{rt}$

$A = 2000e^{0.03(5)}$

Step 3 Use a calculator to solve for A, the total amount.

$A = 2000e^{0.03(5)}$

$A \approx 2323.67$

Use the e^x key on a calculator: $2000e^{(.03*5)} = 2323.668485$

The total amount is $2323.67.

Use the formula $A = Pe^{rt}$ to solve.

7. What is the total amount for an investment of $500 invested at 4.5% and compounded continuously for 10 years?

P = 500 r = 0.045 t = 10

$784.16

8. Randy deposited $1000 into an account that paid 2.8% with continuous compounding. What was her balance after 6 years? $1182.94

9. a. Martin borrows $5500. The rate is set at 6% with continuous compounding. How much does he owe at the end of 2 years? $6201.23

b. Martin found a bank with a better interest rate of 5.5%. How much less does he owe at the end of 2 years? $61.70

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Challenge

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As you have seen, one way to approximate the value of *e* is to let the value of *n* become very large in the sequence of numbers obtained from the expression $1\left(1 + \frac{1}{n}\right)^n$. You can explore some other methods for evaluating *e*.

Consider the sequence 1, $\frac{1}{1}, \frac{1}{2 \cdot 1}, \frac{1}{3 \cdot 2 \cdot 1}, \ldots$

1. Write the 9th term of the sequence. $\frac{1}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

2. Using a calculator, determine

a. the sum of the first 5 terms of the sequence. 2.7083

b. the sum of the first 7 terms of the sequence. 2.718055556

c. the sum of the first 10 terms of the sequence. 2.718281526

3. Use what you know about the value of *e* and the results of Exercise 2 to write an expression for *e* in terms of the given sequence of numbers. $e = 1 + \frac{1}{1} + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2 \cdot 1} + \cdots$

A *continued fraction* is formed by a number added to a fraction whose denominator is a fraction added to a fraction whose denominator is a fraction, and so on, forming a pattern.

- **Example** To evaluate, start with the last denominator.

$$1 + \cfrac{1}{2 + \cfrac{1}{3 + \frac{1}{4}}} = 1 + \cfrac{1}{2 + \cfrac{1}{\frac{13}{4}}} = 1 + \cfrac{1}{2 + \frac{4}{13}} = 1 + \cfrac{1}{\frac{30}{13}} = 1 + \frac{13}{30} = \frac{43}{30}$$

Start.

Complete the continued fraction by finding the missing denominator. Then evaluate.

4. $$2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{\ }}}}}$$

$5 + \frac{5}{6}; \frac{5760}{2119} \approx 2.718263332$

5. $$1 + \cfrac{2}{1 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cfrac{1}{\ }}}}}$$

$18 + \frac{1}{22}; \frac{1,084,483}{398,959} \approx 2.718281828$

6. Continue the pattern further in the fractions above and make an observation.

Possible answer: Both of the given continued fractions can be used to determine the value of *e*.

LESSON 7-6

Problem Solving

The Natural Base, e

Irene reads that the 2004 census of whooping cranes tallied 213 birds at one wildlife refuge in Texas. This number exceeded the 2003 record by 19. If the population of whooping cranes can be modeled using the exponential growth function $P_t = P_0e^{kt}$, the population, P_t, at time t can be found, where P_0 is the initial population and k is the growth factor. Predict the population of whooping cranes over the next few years.

1. What was the size of the population of whooping cranes in 2003? 194

2. Use the population figures for 2003 and 2004 to find the growth factor, k.

$k = 0.0934$

3. Complete the table to predict the population of whooping cranes through 2010.

Year	2006	2007	2008	2009	2010
t	3	4	5	6	7
Population, P_t	257	282	309	340	373

Choose the letter for the best answer.

4. Irene wants to know when the population of whooping cranes will exceed 1000. Using the 2003 population as P_0, which year is the best prediction?

A 2017

B 2019

(C) 2021

D 2023

5. Irene wonders how the 2010 whooping crane population would change if the growth factor doubled. Which statement is true?

A The population would increase by a factor of e^2.

B The population would increase by a factor of $e^{0.0934}$.

(C) The population would increase by a factor of $e^{(0.0934)(7)}$.

D The population would increase by a factor of $7e^2$.

6. How long will it take for an investment in an account paying 6% compounded continuously to double?

A 10.2 years

B 10.8 years

(C) 11.6 years

D 12.4 years

7. Darlene has a sample of a fossil that has 33% of its original carbon-14. Carbon-14 has a half-life of 5730 years. The decay constant for carbon-14 is 1.2×10^{-4}. Find the age of the fossil.

A About 7820 years

B About 8450 years

C About 8980 years

(D) About 9240 years

LESSON 7-6

Reading Strategies

Use a Graphic Organizer

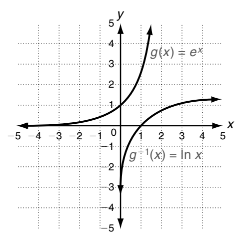
Definition	Facts
The number *e* is an irrational constant like π. You can estimate *e* by using very large values of *n* in the formula. $f(n) = \left(1 + \frac{1}{n}\right)^n$ $\approx 2.7182818\ldots$ $e \approx 2.7182818\ldots$	A logarithm with base e is called a natural logarithm (ln *x*). The functions e^x and ln *x* have the same properties as the other exponential and logarithmic functions you have studied.
Example (compound interest)	**Useful Hints**
For a **principal** investment of $100 with a growth **rate** of 5% for 10 **years** compounded continuously, the total **amount** will be: $A = Pe^{rt}$ $= 100 \bullet e^{0.05 \times 10}$ $= \$164.87$	You can use the property of inverse functions to solve many problems containing *e* and ln. For example: $\ln e^3 = 3$ and $e^{\ln 3} = 3$

Answer each question.

1. a. Rewrite $e^{3 \ln x}$ using the Power Property of logarithms. $e^{3 \ln x} = e^{\ln x^3}$

b. Now simplify. x^3

2. The graph shows $g(x) = e^x$ and $g^{-1}(x) = \ln x$.



a. Label each curve with the correct function.

b. What transformation is represented by the 2 curves?

Reflection over the line $y = x$

c. Explain how you can tell that they are inverse functions.

Possible answer: The *x*- and *y*-values of each point in one graph are reversed in the other graph.

3. $A = Pe^{rt}$ is a formula used for continuously compounded interest.

a. Which variable represents the principal or starting amount? P

b. Which variable represents the time length of the investment? t

c. Which variable represents the rate of interest paid on the investment? r