# Estimating Solutions from Square Root Functions

# 7.8

**FOCUSING QUESTION** How can you approximate a solution to a square root equation using tables or graphs?

## LEARNING OUTCOMES

- I can use tables and graphs to approximate solutions involving square root functions that model real-world problems.
- I can create and use tables and graphs to organize, record, and communicate ideas about square root functions and their related equations.

## ENGAGE

The Portland Head Light, in Cape Elizabeth, Maine, is a lighthouse that is 80 feet tall. In clear air, you can calculate the distance, in miles, that the light can be seen from a boat on the water using the function, $d(x) = 1.32\sqrt{x}$, where $x$ represents the height of the lighthouse in feet. What is the farthest from the lighthouse that a boat could be and still see the light from the lighthouse?

Portland Head Light, Cape Elizabeth, Maine

***$d(80) = 1.32\sqrt{80} \approx 11.8$ miles***

## EXPLORE

The distance that the light from a lighthouse can be seen from a boat on the water is a function of the height of both the lighthouse and the viewer above the water level. If a sailor is viewing the lighthouse from the deck of a ship 15 feet above the water, the maximum distance that he can see a light from a lighthouse on the horizon is calculated using the function $d(h) = \sqrt{\frac{7(h+15)}{4}}$, where $d(h)$ is the maximum distance in miles and $h$ is the height of the lighthouse, in feet, above the water level.

1. In the 1850s, planning began for a lighthouse at Bolivar Point, Texas. Local mariners determined that the light from the lighthouse should be visible from a boat 14 miles offshore. Write an equation that could be used to determine required height of the lighthouse.

   $14 = \sqrt{\frac{7(h+15)}{4}}$



### TEKS

**AR.6A** Estimate a reasonable input value that results in a given output value for a given function, including quadratic, rational, and exponential functions.

**AR.6C** Approximate solutions to equations arising from questions asked about exponential, logarithmic, square root, and cubic functions that model real-world applications tabularly and graphically.

### MATHEMATICAL PROCESS SPOTLIGHT

**AR.1E** Create and use representations to organize, record, and communicate mathematical ideas.

### ELPS

**4C** Develop basic sight vocabulary, derive meaning of environmental print, and comprehend English vocabulary and language structures used routinely in written classroom materials.

### VOCABULARY

radicand, square root, period, pendulum

### MATERIALS

- graphing technology

**RELECT ANSWERS:**

*The inverse operation of taking the square root is squaring a number, so square both members of the equation and then solve the resulting linear or quadratic equation.*

*Approximating the solution to any equation from a table uses the same process of locating a given output value in the table and then identifying its paired input value.*

*If the square root equation is related to a square root function, then it will only have one solution since the square root function values are either always increasing or always decreasing. If the square root equation includes ±f(x), where f(x) is a related square root function, then the equation will have one solution if it is the vertex of graph of the combined functions, +f(x) and –f(x). Otherwise, the equation will have two solutions, one on the graph or table of +f(x) and one on the graph or table of –f(x).*

**2.** Use graphing technology or paper and pencil to generate a table of values like the one shown.

| HEIGHT (FEET) | 95 | 96 | 97 | 98 | 99 | 100 |
|---|---|---|---|---|---|---|
| DISTANCE (MILES) | *13.874* | *13.937* | *14* | *14.062* | *14.124* | *14.186* |

**3.** Use the table to approximate a solution to your equation for the required height of the Bolivar Point lighthouse.
***See margin.***

**4.** The lighthouse that was eventually constructed was 116 feet tall. How much further offshore could a mariner on a boat see the completed lighthouse than the one that was originally needed?
***See margin.***

**5.** At Port Isabel, Texas, ships need to see the lighthouse about $11\frac{1}{4}$ miles offshore. Write an equation that could be used to determine required height of the lighthouse.

$$11\frac{1}{4} = \sqrt{\frac{7(h+15)}{4}}$$

**6.** Use a graph to approximate a solution to the equation. Explain how you used your graph to determine your estimate.
***See margin.***

## REFLECT

- **How could you use inverse operations to solve a square root equation?**
  ***See margin.***
- **How is using a table to approximate the solution to a square root equation similar to using tables to approximate solutions to other types of equations?**
  ***See margin.***
- **How many solutions will a square root equation have? Justify your answer using appropriate mathematical terminology.**
  ***See margin.***



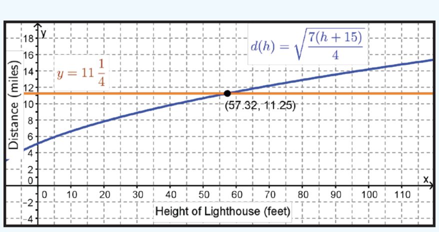
**3.**

| HEIGHT (FEET) | 95 | 96 | 97 | 98 | 99 | 100 |
|---|---|---|---|---|---|---|
| DISTANCE (MILES) | *13.874* | *13.937* | *14* | *14.062* | *14.124* | *14.186* |

*When h = 97, d(h) = 14, so d(97) = 14.*
*To be visible 14 miles offshore, the lighthouse should be 97 feet tall.*

**4.** $d(116) = \sqrt{\frac{7(116+15)}{4}} = \sqrt{229.25} \approx 15.14$ *miles*

*15.14 – 14 = 1.14 miles farther*




**6.** *h ≈ 57 Possible process: Graph both d(h) and $y = 11\frac{1}{4}$ on a graphing calculator. Adjust the window so that you can see the point of intersection. Use the calculator to determine the coordinates of the point of intersection to be approximately (57.32, 11.25). These coordinates mean that when h ≈ 57.32, d(h) = 11.25, so the solution to the equation is approximately 57.*

# EXPLAIN

Square root functions are the inverses of related quadratic functions with restrictions on the domain of the quadratic function that was used to generate it. You can use square root functions to write related equations, then use graphs and tables to approximate solutions to those equations for the value of the independent variable that generates a particular value of the dependent variable.

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## APPROXIMATING SOLUTIONS TO EQUATIONS GRAPHICALLY

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A pendulum clock uses a pendulum to keep time. The amount of time that it takes for a pendulum to swing from one end of its arc to the other and back again is called the **period**. The period of the pendulum, $T(x)$, can be found using the function $T(x) = 2\pi\sqrt{\frac{x}{9.8}}$, where $x$ represents the length of the pendulum in meters and $T(x)$ is the period of the pendulum in seconds.

Mrs. Crick is a clockmaker and wants to make a clock with a pendulum that has a period of one second. How long should the pendulum be?

**ELPS Strategy**

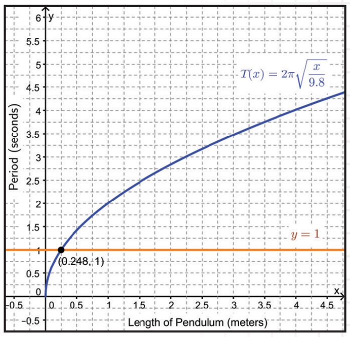
Work with a partner as you read this section. Read through the problem to analyze the information and develop a plan for solving the problem. Look for specific English language structures, such as prepositional phrases and clauses to provide important information and clarification about the clock and its attributes. In your math notebook, answer the following questions.

- What part of the clock does the period describe?
- Which attributes of the pendulum are described by the function $T(x)$?

The period of 1 second is a value of the dependent variable, so use that along with $T(x)$ to write an equation.

$$1 = 2\pi\sqrt{\frac{x}{9.8}}$$

Use graphing technology to graph $T(x)$ and the line $y = 1$ to determine the point on the graph of $T(x)$ with a function value of 1.




The point (0.248, 1) is the intersection point of $T(x)$ and $y = 1$. Notice that 0.248 is a rounded value for $T(x)$. You can approximate that $T(0.248) = 1$ and estimate that when $x = 0.248$ meters, or 24.8 centimeters, the period of the clock that Mrs. Crick is building will be about 1 second. So the pendulum must be 24.8 centimeters long.



## SUPPORTING ENGLISH LANGUAGE LEARNERS

English language structures, such as prepositional phrases and dependent clauses, help students to identify clarifying information about a noun in a sentence. For example, information in Mrs. Crick's clock problem is presented with such clarifying language structures. If students comprehend these structures that are used routinely in written classroom materials (**ELPS 4C**), then they are better able to analyze information presented in a problem and use that information to develop a plan for solving the problem.

## INTEGRATING TECHNOLOGY

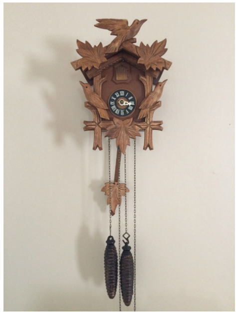
You can use a graphing calculator or app to estimate the solution of a square root equation graphically. Graph the function in $Y1$ and then graph $Y2 = n$, where $n$ is the given output value or function value. Use the calculator's or app's features to determine where the two graphs intersect. Usually, for a square root function, the $x$-coordinate, which represents the solution to the equation, is a rounded value. Hence, the $x$-coordinate of the point of intersection is an approximation for the solution to the equation.

## INTEGRATING TECHNOLOGY

You can use a graphing calculator or app to solve an equation tabularly. Enter the function into $Y1$ of the function editor. In the table feature, scroll up and down until you see the given output value or function value. If you do not see the exact function value, then change the $x$-interval to a smaller number (e.g., 0.1 instead of 1) and look again. The $x$-value in the same row as the function value is the solution to the equation. For square root functions, start out with an $x$-interval of about 1, and scroll until you find dependent variable values that are close to the given value for your equation. Refine the $x$-interval to 0.1, 0.01, or 0.001 to get a closer approximation for your solution.

## APPROXIMATING SOLUTIONS TO EQUATIONS TABULARLY

One of Mrs. Crick's favorite clocks is a cuckoo clock that she brought back from a family vacation in Switzerland.



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Mrs. Crick timed the period of this cuckoo clock to be 1.25 seconds. How long is the pendulum of the cuckoo clock?

To solve this problem, write an equation from $T(x)$.

$$1.25 = 2\pi\sqrt{\frac{x}{9.8}}$$

Use graphing technology to make a table of values for $T(x)$. Look in the column for the dependent variable with a value of 1.25. You may need to refine the interval for $\Delta x$ to see additional rational number values.

| LENGTH (METERS), $x$ | PERIOD (SECONDS), $T(x)$ |
|---|---|
| 0.385 | 1.2454 |
| 0.386 | 1.2470 |
| 0.387 | 1.2486 |
| 0.388 | 1.2502 |
| 0.389 | 1.2518 |
| 0.390 | 1.2534 |
| 0.391 | 1.2550 |

In this case, there is no exact value for the period of 1.25 seconds. However, there are two very close values that round to 1.25: 1.2486 and 1.2502. Use these period values and their corresponding length values to approximate a solution to this equation.

$$1.2486 < 1.25 < 1.2502$$
$$0.387 < x < 0.388$$

The length of the pendulum is between 0.387 meters and 0.388 meters, or approximately 0.3875 meters, which is about 38.75 centimeters.

### APPROXIMATING SOLUTIONS TO EQUATIONS RELATED TO SQUARE ROOT FUNCTIONS

A square root function is the inverse of a quadratic function. In order for the square root relationship to be a function, there is a domain restriction on the related quadratic function that results in a range restriction for the square root function.

An equation that is related to a given function, $f(x)$, is one in which the value of the dependent variable is known and you need to determine the value(s) of the independent variable that generates it. For a square root function, there will only be one pair of values for which this is true.

- Graphically, locate a point on the graph of $f(x)$ that has a $y$-coordinate equal to the given function value. The $x$-coordinate of this point is the $x$-value paired with that function value. This $x$-value is the solution to the equation.
- Tabularly, locate the function value in the dependent variable column or row. The value in the independent variable column or row associated with this function value is the solution to the equation.



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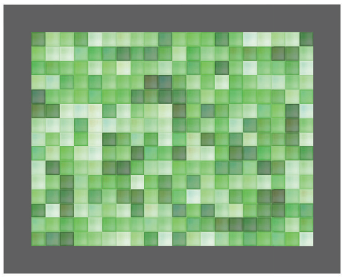
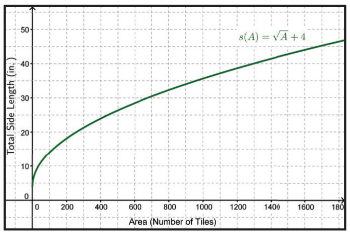
## EXAMPLE 1

An artist makes mosaics with tiles for kitchens and bathrooms. In one kitchen, she needs to cover the wall area behind a stove 40 inches wide with a square mosaic. The pattern will consist of 1-inch square tiles and the frame will be 2 inches wide all around the mosaic. How many tiles will it take to make the correct size mosaic for the stove's width? She can use the function $s(A) = \sqrt{A} + 4$, where s is the total side length of the mosaic including the frame and $A$ is the area of the mosaic itself, to determine how many tiles the pattern will take.



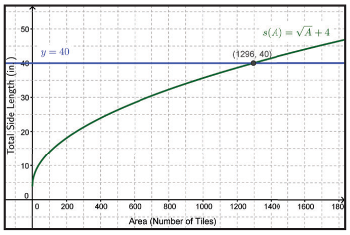
**STEP 1** Graph the function $s(A) = \sqrt{A} + 4$.




**STEP 2** Write an equation related to the function with the required side length, $s$, of 40 inches.

$$40 = \sqrt{A} + 4$$

**STEP 3** Graph $y = 40$ on the same grid with the function and use the coordinates of the intersection point to answer the question, "How many tiles will the pattern take?"




***The intersection point is (1296, 40). So the artist will use 1,296 tiles to make the pattern.***

## YOU TRY IT! #1

The function $t(h) = \sqrt{\frac{h}{4.9}}$ represents the time, $t$, in seconds that it takes for an object to fall a distance, $h$, in meters, due to the Earth's gravity. Two divers jump off the face of a cliff at different heights but at the same time. Diver #1 dives from an unknown height, and Diver #2 dives from a ledge of several meters higher than the first diver. If Diver #1 hits the water at 2.0203 seconds after jumping, and it takes 0.2832 seconds longer for Diver #2, what were their heights on the cliff? Use a graph to determine your answer.
***See margin.***

## ADDITIONAL EXAMPLE

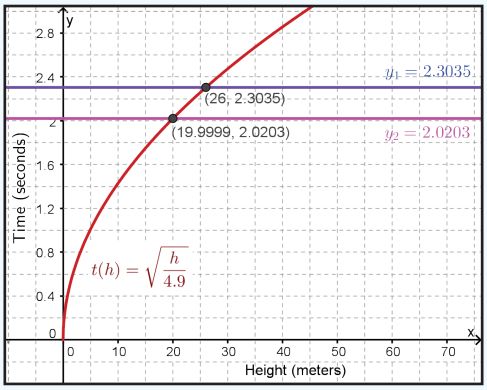
The function $t(h) = \sqrt{\frac{h}{0.811}}$ represents the time, $t$, in seconds it takes for an object to fall a distance, $h$, in meters, on the moon due to gravity. An astronaut has walked on the moon and is doing a repair on his spaceship before preparing to launch to go back to Earth when he drops a wrench. It takes 1.2 seconds to fall to the surface of the moon. How high, in feet, was he holding the wrench when he dropped it? Use a graph to determine your answer.

*Using graphing technology, the graph shows the intersection point for the function $t(h) = \sqrt{\frac{h}{0.811}}$, and the equation $y = 1.2$, is (1.16, 1.2). Since 1.16 is in meters, you must multiply by 3.28 to convert to feet. $1.16 \times 3.28$ is approximately 3.83 feet. The astronaut dropped the wrench from a height of about 3.83 feet.*



**YOU TRY IT! #1 ANSWER:**

*The graph shows intersection points for the function, $t(h) = \sqrt{\frac{h}{4.9}}$, and the equations, $y_1 = 2.0203$ and $y_2 = 2.0203 + 0.2832 = 2.3035$, at (19.9999, 2.0203) and (26, 2.3035) respectively. So Diver #1 jumped from about 20 meters above the water, and Diver #2 jumped from 26 meters above the water.*



## ADDITIONAL EXAMPLE

Use the scenario from Example #2. The container store decides to halve the height of the cylindrical containers making them 12 centimeters tall. This changes the square root function for finding the radius of the cylinder to $r(v) = 0.163\sqrt{V}$. If the radius of the jar stays 5 centimeters, what will the volume of the new jar be? Using the function, a related equation, and a table of values for the function, determine the volume of the newer, shorter jar.

*The equation related to the function is $5 = 0.163\sqrt{V}$. Using graphing technology to create a table of values for the function $r(v) = 0.163\sqrt{V}$, the volume of the jar will be approximately 940 cubic centimeters. This is half the volume of the original jar from Example #2.*

## EXAMPLE 2

Cylindrical jars at a container store are all 24 centimeters tall with different radii. The radius of each jar can be determined based on its volume.

The function for $r$ in terms of volume, with a height of 24 and an approximate value of 3.14 for pi, is $r(V) = 0.115\sqrt{V}$. Using the function, a related equation, and a table of values for the function, determine the volume of a jar with a radius of approximately 5 centimeters.

**STEP 1** Write a related equation for the function $r(V) = 0.115\sqrt{V}$.

$5 = 0.115\sqrt{V}$

**STEP 2** Generate a table of values for the function $r(V) = 0.115\sqrt{V}$ using technology.

| VOLUME, $cm^3$ | 0 | 500 | 1000 | 1500 | 2000 | 2500 | 3000 |
|---|---|---|---|---|---|---|---|
| RADIUS, cm | 0 | 2.57 | 3.64 | 4.45 | 5.14 | 5.75 | 6.30 |

**STEP 3** Since a radius of 5 is between 4.45 and 5.14, generate a new table of input values between 1,500 and 2,000, decreasing the interval of 500 to 100.

| VOLUME, $cm^3$ | 1500 | 1600 | 1700 | 1800 | 1900 | 2000 |
|---|---|---|---|---|---|---|
| RADIUS, cm | 4.45 | 4.60 | 4.74 | 4.88 | 5.01 | 5.14 |

**STEP 4** Interpret the values in the table.

For the function value of 5.01, there is an input value of 1900. A jar with a radius of 5 centimeters will have a volume of approximately 1,900 cubic centimeters.

## YOU TRY IT! #2

Brandy has owned a small clothing boutique since 2009. She has tracked how many customers she has had each year. Her accountant says that if the growth rate of the business stays about the same, the number of customers can be predicted using the function $c(t) = 210 + \sqrt{13{,}000t}$, where $c$ is the number of customers and $t$ is the number of years since 2009. Using the function, a related equation, and a table of values for the function, determine the year during which Brandy can anticipate having 600 customers.

***See margin.***



### YOU TRY IT! #2 ANSWER:

*The equation related to the function is $600 = 210 + \sqrt{13000t}$. According to the table of values for the function $c(t) = 210 + \sqrt{13000t}$, her shop will have 600 customers between the tenth year and the twelfth year, so it will most likely occur during the twelth year, 2020.*

| TIME, IN YEARS | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|
| CUSTOMERS | 512 | 532 | 662 | 571 | 588 | 605 | 621 |

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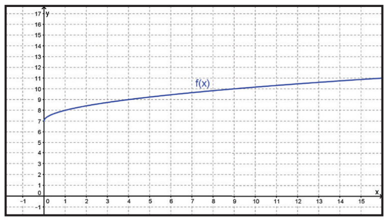
## PRACTICE/HOMEWORK

**1.** The function $f(x) = \sqrt{x} + 7$ is graphed below.




**A.** What is the value of $x$ when $f(x) = 9$? ***x = 4***
**B.** What is the value of $x$ when $f(x) = 7$? ***x = 0***
**C.** What is the value of $x$ when $f(x) = 10.5$? ***x ≈ 12.5 or 12 ≤ x ≤ 13***

**2.** Values from the function $g(x) = 5\sqrt{3x}$ are shown in the table below.

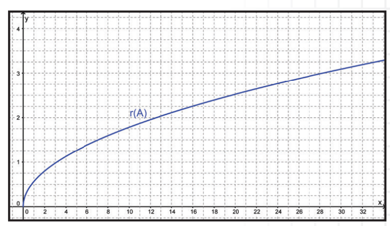
| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $g(x)$ | 0 | 8.7 | 12.2 | 15 | 17.3 | 19.4 | 21.2 |

**A.** What is the value of $x$ when $g(x) = 15$? ***x = 3***
**B.** What is the value of $x$ when $g(x) = 21.2$? ***x = 6***
**C.** What is the value of $x$ when $g(x) = 12$? ***x ≈ 1.9 or 1 ≤ x ≤ 2***

*Use the situation described below to answer questions 3 – 5.*

### GEOMETRY

The radius of a circle, $r(A)$, can be represented by the function $r(A) = \sqrt{\frac{A}{\pi}}$, with $A$ representing the area of the circle. The graph of this function is shown below.




**3.** Write an equation whose solution would give you the area of a circle with a radius of 3 inches.
***$3 = \sqrt{\frac{A}{\pi}}$***

**4.** Reference the given graph of $r(A)$. Determine the approximate point on the graph of $r(A)$ with a function value of 3.
***(28, 3)***

**5.** What does the point mean for the situation?
***When the radius of a circle is 3 inches, the area of the circle is about 28 inches.***

*Use the information below to answer questions 6 – 8.*

## SCIENCE

The function $t(h) = \sqrt{\frac{h}{16.1}}$ represents the time, $t$, in seconds that it takes for an object to fall a distance, $h$, in feet due to the Earth's gravity.

**6.** Bear is hiking in rugged terrain and encounters a cliff that he needs to descend. He has a rope to help with his descent, and wants to make sure it is long enough to make it all the way down the cliff. To help him determine the distance to the ground he throws a rock off the side of the cliff and times how long it takes to reach the ground. Write an equation whose solution would give you the height of the cliff if it takes 1.7 seconds for the rock to reach the ground.

***$1.7 = \sqrt{\frac{h}{16.1}}$***

**7.** Use graphing technology to graph $t(h)$ and the line $y = 1.7$. Determine the point on the graph of $t(h)$ with a function value of 1.7.

***(46.529, 1.7)***

**8.** If Bear has a 52-foot rope, will he have enough to rappel down the cliff? Explain.

***Yes. Since he has 52 feet of rope, and the cliff is about 47 feet high, he should have enough rope.***

*Use the situation described below to answer questions 9 – 12.*

## SCIENCE

The speed of a car when it goes into a skid, $s(d)$, can be represented by the function $s(d) = \sqrt{30fd}$, where $d$ is the length of the skid in feet. The variable $f$ is called the coefficient of friction, and it varies based on the road conditions at the time of the skid. Eli was involved in a car accident on a day when $f$ was determined to be 0.8. The resulting function is $s(d) = \sqrt{24d}$.

**9.** Eli reported that he was going about 40 miles an hour at the time of the accident. Write an equation whose solution would give you the approximate length of the skid if he was going 40 miles per hour.

***$40 = \sqrt{24d}$***

**10.** Reference the table below showing values of $s(d)$. Determine the approximate point of $s(d)$ with a function value of 40.

| LENGTH OF SKID, $d$ | 0 | 6 | 24 | 54 | 67 | 96 | 105 | 149 | 150 | 151 |
|---|---|---|---|---|---|---|---|---|---|---|
| APPROXIMATE SPEED, $s(d)$ | 0 | 12 | 24 | 36 | 40.1 | 48 | 50.2 | 59.8 | 60 | 60.2 |

***Answers will vary slightly, but should be close to (66.7, 40).***

**11.** What does the point mean for the situation?

***The approximate skid length for a car going 40 mph is 66.7 feet.***



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12. If the actual length of the skid was 70 feet, was the speed of the car really 40 mph? Explain.
***No, because the speed would need to be more than 40 mph to create a 70-foot skid.***

*Use the situation described below to answer questions 13 – 15.*

## SCIENCE

Carrie is at an amusement park and is in line for the pendulum ride. The carriage holding the people swings back and forth, moving like a pendulum. The function that represents the time in seconds of one complete swing, $T(x)$, based on the pendulum length, $x$, in meters, is $T(x) = 2\pi\sqrt{\frac{x}{9.8}}$.

13. While in line, Carrie notices that the period of the pendulum is about 10 seconds. Write an equation whose solution would give you the length of the pendulum if it takes 10 seconds to swing back and forth.
$10 = 2\pi\sqrt{\frac{x}{9.8}}$

14. Use graphing technology to make a table of values for $T(x)$. Use the table to determine the approximate point of $T(x)$ with a function value of 10.
***Approximately (25, 10)***

15. What does the intersection point mean for the situation?
***The pendulum has a length of about 25 meters.***

*Use the situation described below to answer questions 16 – 18.*

## GEOMETRY

A set of cylindrical pipes at the hardware store consists of cylinders all having a height of 18 inches but with different radii. The radius of each cylinder can be determined based on its volume, using the function $r(V) = 0.133\sqrt{V}$.

16. Write an equation that would determine the volume of a cylinder having a radius of 2 inches.
$2 = 0.133\sqrt{V}$

17. Generate a table of values for $r(V)$ using technology. Use the table to determine the approximate point of $r(V)$ with a function value of 2.
***Approximately (226, 2)***

18. What does the intersection point mean for the situation?
***A cylinder with a height of 18 inches and a radius of 2 inches has a volume of about 226 in³.***

*For questions 19 – 23, solve using either a graph or table.*

19. Given the function $f(x) = \sqrt{\frac{x}{4}}$, determine the value of $x$ when $f(x) = 1.5$.
$x = 9$

**20.** Given the function $f(x) = 1.5\sqrt{x}$, determine the value of $x$ when $f(x) = 4$.
***Answers should be close to $x \approx 7.1$.***

## GEOMETRY

**21.** A food company is making canned soup. The radius of each can, $r(V)$, can be determined based on its volume, using the function $r(V) = \sqrt{\frac{V}{26}}$.

**A.** Write an equation whose solution would give you the volume of a can with a radius of 3 cm.
$3 = \sqrt{\frac{V}{26}}$

**B.** What is the solution to your equation, and what does it mean for the situation?
***The solution is $V = 234$; the volume of the soup can with a radius of 3 cm is 234 $cm^3$.***

**22.** The equation $t(h) = \sqrt{\frac{h}{16.1}}$ represents the time, *t*, in seconds that it takes for an object to fall a distance, *h*, in feet, due to the Earth's gravity. Liz accidentally dropped her bottled water from the top of the bleachers during a game. It took 1 second to reach the ground.

**A.** Write an equation whose solution would give you the height from which the bottle was dropped.
$1 = \sqrt{\frac{h}{16.1}}$

**B.** What is the solution to your equation, and what does it mean for the situation?
***$h \approx 16$; the height of the bleachers where the water bottle was dropped is approximately 16 feet.***

## SCIENCE

**23.** For a movie scene, a stunt car driver must skid her car to a stop just in front of a building. She will be driving at 60 miles per hour when she applies her brakes. To find the distance, *d*, she should apply the brakes in order to come to a stop before she reaches the wall, she can use the equation $60 = \sqrt{18d}$. How far from the building must she apply the brakes in order to safely stop?
***She must apply the brakes at least 200 feet from the building.***



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