

Estimating Solutions from Logarithmic Functions

7.7



FOCUSING QUESTION How can you approximate a solution to a logarithmic equation using tables or graphs?

LEARNING OUTCOMES

- I can use tables and graphs to approximate solutions to equations involving logarithmic functions that model real-world problems.
- I can create and use tables and graphs to organize, record, and communicate ideas about logarithmic functions and their related equations.

ENGAGE

Earthquake intensity is measured using the Richter scale, which is a logarithmic scale describing how strong an earthquake is as a logarithmic function of the amplitude of the earthquake's seismic waves. Recall that logarithms are based on powers of 10. So, to compare the strengths of earthquakes, subtract their magnitudes and then raise 10 to that power.

The largest earthquake recorded in Texas was a magnitude 5.8 earthquake recorded near Valentine, Texas, in 1931. In 2015, a magnitude 4.0 earthquake was recorded near Mansfield, Texas. How many times greater in amplitude was the Valentine earthquake than the Mansfield earthquake?

5.8 - 4.0 = 1.8, and $10^{1.8} \approx 63.1$, so the Valentine earthquake had 63.1 times the amplitude of the Mansfield earthquake.



San Andreas Fault
Source: U.S. Geological Survey



EXPLORE

The loudness of a sound is measured in decibels (dB). One bel is the unit of measure for the loudness of a sound wave. One decibel is one-tenth of a bel (the prefix *deci-* means "one-tenth"). The loudness of a sound is a function of the pressure generated by the sound wave producing it. For sounds produced in air, the loudness, $L(p)$, can be written as the following function, where p represents the pressure of a sound wave in micropascals and $L(p)$ is in decibels.

$$L(p) = 20 \log\left(\frac{p}{20}\right)$$

The table shows the estimated loudness, in decibels, of some common sounds.

SOUND	INTENSITY (dB)
WHISPER	30
MOSQUITO BUZZING	40
NORMAL CONVERSATION	60
VACUUM CLEANER	70
CITY TRAFFIC	85
JACKHAMMER AT 50 FEET	95
MOTORCYCLE	100
LOUD ROCK CONCERT	115
JET ENGINE AT 100 FEET	140

TEKS

AR.6A Estimate a reasonable input value that results in a given output value for a given function, including quadratic, rational, and exponential functions.

AR.6C Approximate solutions to equations arising from questions asked about exponential, logarithmic, square root, and cubic functions that model real-world applications tabularly and graphically.

MATHEMATICAL PROCESS SPOTLIGHT

AR.1E Create and use representations to organize, record, and communicate mathematical ideas.

ELPS

4F Use visual and contextual support and support from peers and teachers to read grade-appropriate content area text, enhance and confirm understanding, and develop vocabulary, grasp of language structures, and background knowledge needed to comprehend increasingly challenging language.

VOCABULARY

logarithm, base, power, magnitude

MATERIALS

- graphing technology

- Write an equation that could be used to determine p , the sound pressure of a sound wave generated by a motorcycle.

$$100 = 20\log\left(\frac{p}{20}\right)$$

- Use graphing technology or paper and pencil to generate a table of values like the one shown.

SOUND PRESSURE, p (MICROPASCALS)	1,999,900	2,000,000	2,000,100	2,000,200	2,000,300	2,000,400
LOUDNESS, $L(p)$ (DECIBELS)	99.9996	100	100.0004	100.0009	100.0013	100.0017

- Use the table to approximate a solution to your equation for the sound generated by a motorcycle.

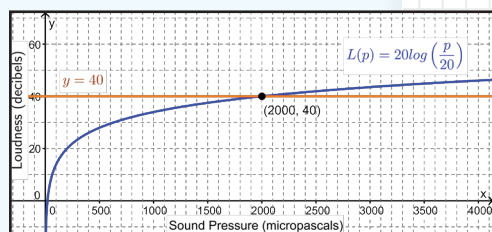
See margin.

- Use your function to write an equation that could be used to determine the sound pressure of a sound wave generated by a mosquito.

$$40 = 20\log\left(\frac{p}{20}\right)$$

- $p = 2000$

Possible process: Graph both $L(p)$ and $y = 40$ on a graphing calculator. Adjust the window so that you can see the point of intersection. Use the calculator to determine the coordinates of the point of intersection to be (2000, 40). These coordinates mean that when $p \approx 2000$, $L(p) = 40$, so the solution to the equation is 2000.



- Use a graph to approximate a solution to the equation. Explain how you used your graph to determine your estimate.

See margin.

- How many times more powerful is the pressure of the sound wave produced by the motorcycle than the sound wave produced by the mosquito?

$$\frac{2,000,000 \text{ micropascals}}{2000 \text{ micropascals}} = 1000 \text{ times}$$

- How many times greater is the loudness of the sound produced by the motorcycle than the sound produced by the mosquito?

$$\frac{100 \text{ dB}}{40 \text{ dB}} = 2.5 \text{ times}$$

- What is the difference between the pressure of the sound wave produced by the motorcycle and the sound wave produced by the mosquito?

$$2,000,000 \text{ micropascals} - 2,000 \text{ micropascals} = 1,998,000 \text{ micropascals}$$

- What is the difference between the loudness of the sound produced by the motorcycle and the sound produced by the mosquito?

$$100 \text{ dB} - 40 \text{ dB} = 60 \text{ dB}$$

-

SOUND PRESSURE, p (MICROPASCALS)	1,999,900	2,000,000	2,000,100	2,000,200	2,000,300	2,000,400
LOUDNESS, $L(p)$ (DECIBELS)	99.9996	100	100.0004	100.0009	100.0013	100.0017

When $p = 2,000,000$, $L(p) = 100$, so $L(2,000,000) = 100$.

The sound generated by a motorcycle has a pressure of 2,000,000 micropascals.



REFLECT

- How does the difference in the sound pressure compare to the amounts of sound pressure for the motorcycle and the mosquito? Why do you think that is the case?
See margin.
- How does the difference in loudness compare to the amounts of loudness for the motorcycle and the mosquito? Why do you think that is the case?
See margin.
- How do logarithms help you compare larger numbers with smaller numbers?
See margin.



EXPLAIN

Logarithmic functions are used to model situations with a wide range of very large or very small numbers. Base-10 logarithms are based on powers of 10, so very large numbers such as 2,150,000 or very small numbers such as 0.00000713 can be written as logarithms. Use base-10 logarithms to write $\log_{10}(2,150,000) \approx 6.332$ and $\log_{10}(0.00000713) \approx -5.147$. Now, you can use 6.332 and -5.147 to compare these numbers much more easily.

There are two very common bases that are used with logarithms: 10 and e . Base-10 logarithms are written without the 10 as a base using $\log(x)$ notation. Base- e logarithms are also called **natural logarithms** and written using $\ln(x)$ notation.

You can use logarithmic functions to write related equations, then use graphs and tables to approximate solutions to those equations for the value of the independent variable that generates a particular value of the dependent variable.

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APPROXIMATING SOLUTIONS TO EQUATIONS GRAPHICALLY

Internet search engines use a page rank algorithm to describe how important a web page is compared to other web pages. The algorithm is based on a logarithmic function and ranks pages from 0 to 10 based on the number of pages that link to the given web page. One search engine uses the function $r(x) = \log(1.2x)$, where x is the number of pages linked to the given web page, to determine a page rank.

Melinda runs a blog with a page rank of 2.3. How many pages have links to her blog? The page rank, 2.3, is a value of the dependent variable, so use that along with $r(x)$ to write an equation.

$$2.3 = \log(1.2x)$$

REFLECT ANSWERS:

The difference in sound pressures is almost the same as the sound pressure of the motorcycle because the sound pressure produced by the mosquito is so much smaller than the sound pressure produced by the motorcycle.

The difference in loudness is in between the loudnesses of the motorcycle and the mosquito, so there is a noticeable difference between the two loudnesses. You notice the difference because the magnitudes of each loudness are close in size to each other (e.g., tens compared to hundreds instead of millions compared to thousands).

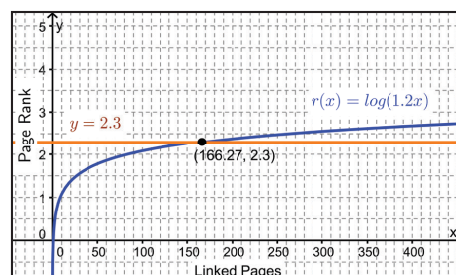
Logarithms take into account the power of 10 represented by the number, which is very close to the number of digits contained in the number. Using powers of 10 to compare numbers makes it easier to think about the relationship between a smaller number and a much larger number.

INTEGRATING TECHNOLOGY

You can use a graphing calculator or app to estimate the solution of a logarithmic equation graphically. Graph the function in $Y1$ and then graph $Y2 = n$, where n is the given output value or function value. Use the calculator's or app's features to determine where the two graphs intersect. Usually, for a logarithmic function, the x -coordinate, which represents the solution to the equation, is a rounded value. Hence, the x -coordinate of the point of intersection is an approximation for the solution to the equation. Also, for a logarithmic function, you may have to set the x -axis to a very large number in order to see the desired output value.

Use graphing technology to graph $r(x)$ and the line $y = 2.3$ to determine the point on the graph of $r(x)$ with a function value of 2.3.

The point $(166.27, 2.3)$ is the intersection point of $r(x)$ and $y = 2.3$. Notice that 166.27 is a rounded value for t . You can approximate that $r(166.27) = 2.3$ and estimate that Melinda's blog has approximately 166 pages that link to it, to give it a page rank of 2.3.



APPROXIMATING SOLUTIONS TO EQUATIONS TABULARLY

ELPS Strategy

Work with a partner as you read this section. Read through the problem and the problem-solving model used to determine a solution. As you do so, ask your partner or your peer for support as needed to look for language structures that you need to make sense of both the problem and its solution. Ask your partner or teacher:

- How does the sentence structure explain the solution process?
- In each sentence, what are the subject, verb, object, and other descriptive phrases?

Melinda notices that her blog has a page rank of 2.3 while a competing blog has a page rank of 5.2. How many more websites link to the competing blog than link to Melinda's blog?

You can use a problem-solving model to answer this question.

ANALYZE THE GIVEN INFORMATION

- Melinda's blog has a page rank of 2.3.
- The competing blog has a page rank of 5.2.
- Page rank is calculated using the function $r(x) = \log(1.2x)$.

DETERMINE A SOLUTION

Use two tables of values for $r(x)$. One with $r(x)$ around 2.3 and one with $r(x)$ around 5.2.

NUMBER OF LINKS, x	PAGE RANK, $r(x)$	NUMBER OF LINKS, x	PAGE RANK, $r(x)$
165	2.2967	132070	5.19999
166	2.2993	132071	5.19999
167	2.3019	132072	5.19999
168	2.3045	132073	5.20000
169	2.3071	132074	5.20000
170	2.3096	132075	5.20000
171	2.3122	132076	5.20001

Melinda's blog, with a page rank of 2.3, has about 167 web pages that link to it. The competing blog, with a page rank of 5.2, has about 132,072 web pages that link to it.

$$132,073 - 166 = 131,907$$

The competing blog has about 131,907 more web pages that link to it than does Melinda's blog.

FORMULATE A PLAN OR STRATEGY

- Use a table to approximate the number of website links to each blog and then calculate the difference.

JUSTIFY THE SOLUTION

In the function $r(x)$, the page rank is the dependent variable and is a function of the number of web pages that link to the blog. The problem asks for the number of web pages that link to the blog, which is the independent variable. Thus, you need to set up and solve, or at least approximate the solution to, an equation for Melinda's blog and for the competing blog.

EVALUATE THE REASONABLENESS OF THE SOLUTION

Logarithms are a way of comparing very large numbers with each other or with smaller numbers. A page rank of 5.2 corresponds loosely with $10^{5.2}$, which is about 158,000. A page rank of 2.3 corresponds loosely with $10^{2.3}$, which is about 200. Since 158,000 is much greater than 200, subtracting 200 will not make the difference much less than 158,000. The calculated solution of 131,907 is reasonably close to the estimate of 158,000, so the solution is reasonable.

APPROXIMATING SOLUTIONS TO EQUATIONS RELATED TO LOGARITHMIC FUNCTIONS

A logarithmic function is the inverse of an exponential function. Some logarithms that are used the most often are base-10, base-2, or base- e . An equation that is related to a given function, $f(x)$, is one in which the value of the dependent variable is known and you need to determine the value(s) of the independent variable that generates it. For a logarithmic function, there will only be one pair of values for which this is true.

- Graphically, locate a point on the graph of $f(x)$ that has a y -coordinate equal to the given function value. The x -coordinate of this point is the x -value paired with that function value. This x -value is the solution to the equation.
- Tabularly, locate the function value in the dependent variable column or row. The value in the independent variable column or row associated with this function value is the solution to the equation.

**INTEGRATING TECHNOLOGY**

You can use a graphing calculator or app to solve an equation tabularly. Enter the function into Y1 of the function editor. In the table feature, scroll up and down until you see the given output value or function value. If you do not see the exact function value, then change the x -interval to a smaller number (e.g., 0.1 instead of 1), and look again. The x -value in the same row as the function value is the solution to the equation. For logarithmic functions, you may wish to use the table set feature so that you can input x -values instead of scrolling.

ADDITIONAL EXAMPLE

Use the logarithmic function from **YOU TRY IT #1**. The band director is allowed to hire an assistant director when the band reaches 350 students. He is hoping this will happen within 5 years. Write a related equation, and use a graph to determine the number of band members the band will have after 5 years. Then answer the question based on your findings.

The equation whose solution yields an answer to the question is $5 = \frac{\log x - \log 50}{\log 1.5}$. Using graphing technology, you can see that the band has 380 members at the 5-year mark. Further exploring the graph, in 4 years, the band will only have 253 members. If the director has to wait for 350 members in order to hire an assistant director, then he will have to wait for the 5th year to be allowed to hire.



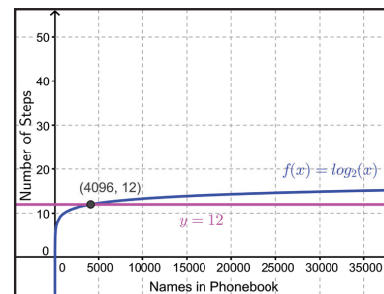
EXAMPLE 1

Some students used the function $f(x) = \log_2 x$ to test the theory that you can find a name in a phone book in a certain number of steps $f(x)$ based on the number of names, x , contained in the phone book. After several trials, the students came up with an average of 12 steps to find some randomly selected names in their hometown phone book. Graph the function and then write an equation to determine a function value of 12 on the graph of $f(x) = \log_2 x$. Using the intersection point, determine approximately how many names the phone book contains.

STEP 1 Write an equation for the average of 12 steps; then graph it and the function $f(x)$ on the same grid.

The line $y = 12$ is shown on the graph with the function $f(x)$.

STEP 2 Find the coordinates of point B , the intersection of the graphs for $f(x)$ and $y = 12$.



The intersection point, B , has an x -coordinate that is slightly greater than 4,000 and a y -coordinate 12.

STEP 3 Interpret the meaning of the coordinates.

You know that the y -coordinate is the number of steps, on average, that it took the students to find the names. The x -coordinate indicates the total number of names in the phone book, which is slightly greater than 4,000. There are slightly more than 4,000 names in the phone book.



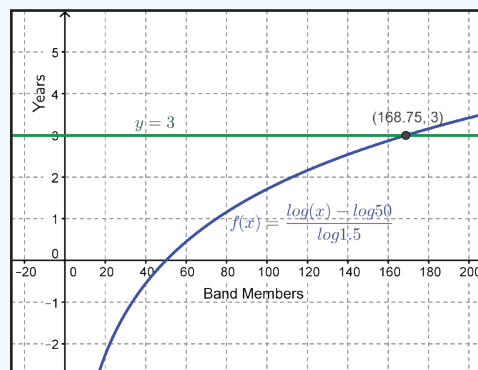
YOU TRY IT! #1

A high school band has 50 band members this year. Their director has a goal of increasing the band by 50% each year. The function $f(x) = \frac{\log x - \log 50}{\log 1.5}$ represents the number of years it will take to have x students in the band at a growth rate of 1.5. Write a related equation, and use a graph to determine the number of members the band will have after 3 years.

See margin.

YOU TRY IT! #1 ANSWER:

$3 = \frac{\log x - \log 50}{\log 1.5}$. The intersection point of the graphs of $y = 3$ and $f(x) = \frac{\log x - \log 50}{\log 1.5}$ indicates that there will be approximately 170 band members after 3 years.





EXAMPLE 2

The level of sound, D , in decibels is defined as the function $D(I) = 10 \log\left(\frac{I}{10^{-16}}\right)$ where I is the sound intensity in watts per square centimeter. Determine the sound intensity of a hair dryer advertised as “quiet” with a sound level of 64 decibels. Write an equation for 64 decibels related to the function $D(I)$ and use the table to approximate the sound intensity of the hair dryer.

SOUND INTENSITY $\left(\frac{W}{cm^2}\right)$	3.162×10^{-11}	1×10^{-10}	3.162×10^{-10}	1×10^{-9}	3.162×10^{-9}	1×10^{-8}
SOUND LEVEL (dB)	55	60	65	70	75	80

STEP 1 Write an equation for the hair dryer with an advertised sound level of 64 decibels.

The equation $64 = 10 \log\left(\frac{I}{10^{-16}}\right)$ will give you the answer to the question, “What is the sound intensity of a hair dryer advertised as ‘quiet’ with a sound level of 64 decibels?”

STEP 2 Find the output value of 64 in the table of values.

The sound level of 64 decibels is not shown in the table. However, a very close value, 65 decibels, is included.

STEP 3 Find a sound intensity related with 65 decibels in the table.

The input value 3.162×10^{-10} is associated with the output value of 65.

STEP 4 Use graphing technology with a table to determine the independent value of the function $D(I)$ that corresponds more closely with a dependent value of 64.

Using graphing technology, you can change the intervals between successive independent values in the table to tenths to determine that the dependent value 64 corresponds to an independent function value of 2.512×10^{-10} . Therefore, $D(2.512 \times 10^{-10}) \approx 64$.

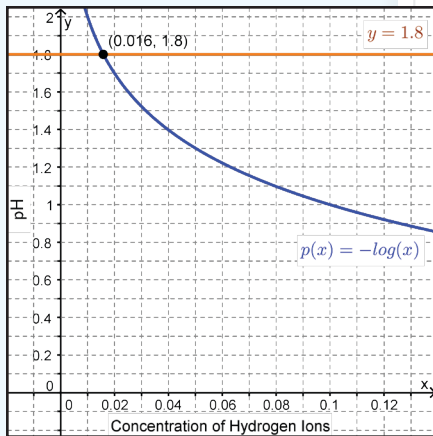
A hair dryer advertising a sound level of 64 decibels has a sound intensity of 2.512×10^{-10} watts per square centimeter.

ADDITIONAL EXAMPLE

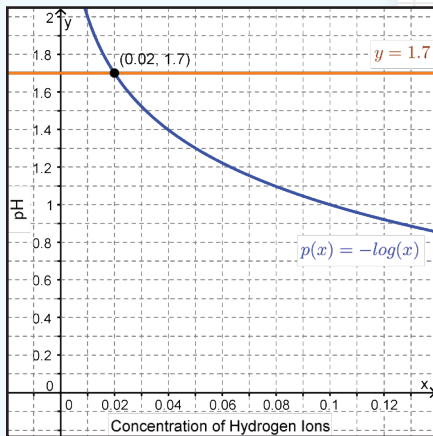
Use graphing technology to create a table of values for $f(x) = \frac{1}{0.75}(\ln x + \ln 150)$. What is the approximate solution for x when $f(x)$ is 11? Write a related equation, and answer the question based on your findings.

The equation whose solution yields an answer to the question is $11 = \frac{1}{0.75}(\ln x + \ln 150)$. Using graphing technology, you can find that $f(x) = 11$ when x is approximately 25.5.

2. $x = 0.016$



4. $x = 0.02$



YOU TRY IT! #2

In Biology class, a sample of bacteria was contaminated. After approximately 7 hours, the sample had reduced to approximately 2,000 bacteria. The students wanted to find out how many bacteria were included in the sample before the contamination. They found a function that models a declining population, $t(x) = \frac{1}{0.25}(\ln x - \ln(2000))$ where $t(x)$ is the time in hours that the bacteria declined, the rate of decline is 0.25, and x is the original population size of the sample of bacteria. Write an equation related to the function that would show the original population size seven hours after the contamination. Then use a table of function values to estimate how many bacteria were in the original sample.

See margin.



PRACTICE/HOMEWORK

Use the scenario below to complete questions 1–7.



SCIENCE

The acidity of a liquid is called the pH of the liquid. This is based on the concentration of hydrogen ions, x , in the liquid. The formula for calculating the pH of a liquid, $p(x)$, is shown.

$$p(x) = -\log x$$

The table shows the pH of various liquids.

LIQUID	pH
LEMON JUICE	1.8
LIME JUICE	1.7
VINEGAR	2.4

- Write an equation that can be used to determine the concentration of hydrogen ions, x , in lemon juice.
 $1.8 = -\log x$
- Use graphing technology to plot the pH function, $p(x)$, and $y = 1.8$. Use the intersection feature to determine the concentration of hydrogen ions, x , in lemon juice.
See margin.
- Write an equation that can be used to determine the concentration of hydrogen ions, x , in lime juice.
 $1.7 = -\log x$
- Use graphing technology to plot the pH function, $p(x)$, and $y = 1.7$. Use the intersection feature to determine the concentration of hydrogen ions, x , in lime juice.
See margin.

YOU TRY IT! #2 ANSWER:

The related equation is $7 = \frac{1}{0.25}(\ln x - \ln(2000))$. In the table, the dependent value of approximately 7 is paired with the independent value of 12,000. If the rate of decline was estimated correctly, the original sample contained approximately 12,000 bacteria.

x	2,000	4,000	8,000	12,000	16,000	20,000	24,000
$t(x)$, IN HOURS	0	2.77	5.55	7.17	8.32	9.21	9.94

5. Write an equation that can be used to determine the concentration of hydrogen ions, x , in vinegar.

$$2.4 = -\log x$$

6. Use a graphing calculator to plot the pH function, $p(x)$, and $y = 2.4$. Use the intersection feature to determine the concentration of hydrogen ions, x , in vinegar.
See margin.

7. Use your answers to previous questions to determine how many times greater the concentration of hydrogen ions present in lime juice is than the hydrogen ions present in lemon juice.

$$\frac{0.02}{0.016} = 1.25 \text{ times greater}$$

Use the scenario below to complete questions 8 – 14.



SCIENCE

The loudness of a sound is measured in decibels (dB). One bel is the unit of measure for the loudness of a sound wave. One decibel is one-tenth of a bel (the prefix *deci-* means “one-tenth”). The loudness of a sound is a function of the pressure generated by the sound wave producing it. For sounds produced in air, the loudness, $L(p)$, can be written as the following function, where p represents the pressure of a sound wave in micropascals and $L(p)$ is in decibels.

$$L(p) = 20\log\left(\frac{p}{20}\right)$$

The table shows the estimated loudness, in decibels, of some common sounds.

SOUND	LOUDNESS (DECIBELS) $L(p)$
GARBAGE DISPOSAL	80
GARBAGE TRUCK	100
POWER SAW	104

The table below shows the loudness in decibels, $L(p)$, based on the pressure generated by the sound wave producing it.

p	200,000	355,656	632,456	1,124,683	2,000,000	3,556,559
$L(p)$	80	85	90	95	100	105

8. Write an equation that could be used to determine p , the sound pressure generated by the sound wave of a garbage disposal.

$$80 = 20\log\left(\frac{p}{20}\right)$$

9. Use the table to approximate a solution to your equation for the sound generated by a garbage disposal.

$$P = 200,000 \text{ micropascals}$$

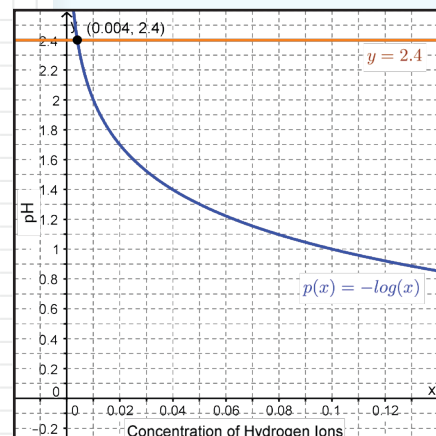
10. Write an equation that could be used to determine p , the sound pressure generated by the sound wave of a garbage truck.

$$100 = 20\log\left(\frac{p}{20}\right)$$

11. Use the table to approximate a solution to your equation for the sound generated by a garbage truck.

$$P = 2,000,000 \text{ micropascals}$$

6. $x = 0.004$



12. Write an equation that could be used to determine p , the sound pressure generated by the sound wave of a power saw.
 $104 = 20\log\left(\frac{p}{20}\right)$
13. Enter the function $L(p)$ into graphing technology. Use the table feature to approximate a solution to your equation for the sound generated by a power saw.
 p is approximately 3,169,787 micropascals (answers may vary)
14. How many times greater is the sound pressure created by a garbage truck than the sound pressure created by a garbage disposal?
 $\frac{2,000,000}{200,000} = 10$ times greater

Use the scenario below to complete questions 15 – 20.

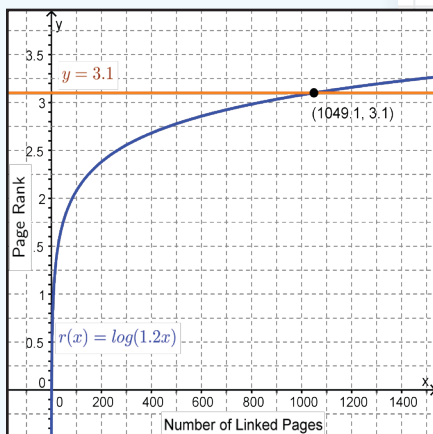


TECHNOLOGY

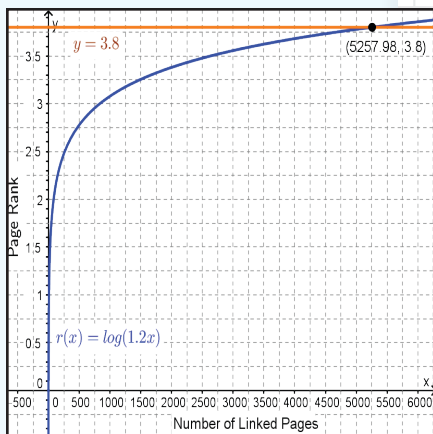
Internet search engines use a page rank algorithm to describe how important a web page is compared to other web pages. The algorithm is based on a logarithmic function and ranks pages from 0 to 10 based on the number of pages that link to the given web page. One search engine uses the function shown below where x is the number of pages linked to the given web page, to determine a page rank.

$$r(x) = \log(1.2x)$$

16. Approximately 1049 pages are linked to Kevin's webpage.



17. Approximately 5258 pages are linked to Kevin's webpage.



15. Kevin has a webpage for his band, Orion, Arise. After 6 months, the webpage had a page rank of 3.1. Write an equation that could be used to determine the number of pages linked to his band's webpage.
 $3.1 = \log(1.2x)$
16. Graph both $r(x)$ and $y = 3.1$ using graphing technology. Use the intersection feature to determine an approximation of pages linked to Kevin's webpage.
See margin.
17. About 1 year after creating the webpage for his band, the page rank is 3.8. Use a graphing calculator and its intersection feature to determine an approximation of pages linked to Kevin's webpage.
See margin.
18. What was the difference in his page rank from 6 months after creating the webpage to 1 year after creating the webpage?
 $3.8 - 3.1 = 0.7$
19. Which statement describes the relationship between the number of pages linked to Kevin's webpage at 6 months and again at 1 year?
A. The number of pages linked to his webpage after 1 year is about 5 times the number of pages linked to his webpage after 6 months.
B. The number of pages linked to his webpage after 6 months is about 5 times the number of pages linked to his webpage after 1 year.
20. How many more webpages were linked to Kevin's webpage at 1 year than at 6 months?
Approximately 4209 pages