## **TEKS**

AR.6A Estimate a reasonable input value that results in a given output value for a given function, including quadratic, rational, and exponential functions.

AR.6C Approximate solutions to equations arising from questions asked about exponential, logarithmic, square root, and cubic functions that model real-world applications tabularly and graphically.

#### MATHEMATICAL **PROCESS SPOTLIGHT**

AR.1E Create and use representations to organize, record, and communicate mathematical ideas.

## **ELPS**

3C Speak using a variety of grammatical structures, sentence lengths, sentence types, and connecting words with increasing accuracy and ease as more English is acquired.

## VOCABULARY

solution, estimate, exponential, base, power, semiannual, continuous, discrete

#### **MATERIALS**

graphing technology

# **Estimating Solutions from** 7.6 **Exponential Functions**



**FOCUSING QUESTION** How can you approximate a solution to an exponential equation using tables or graphs?

#### LEARNING OUTCOMES

- I can use tables and graphs to approximate solutions to equations involving exponential functions that model real-world problems.
- I can create and use tables and graphs to organize, record, and communicate ideas about exponential functions and their related equations.

# **ENGAGE**

A colony of sponges in the Gulf of Mexico increases in size by a rate of 10% each year. If the area of the colony was 14.5 acres when it was first discovered, write a function, f(t), that describes the size of the colony t years after it was discovered. Assume that no sponges were harvested during this time.



Sponge Docks, Tarpon Springs, Florida

#### f(t) = 14.5(1.10)<sup>t</sup>

# **EXPLORE**

1.

Angelica received a \$7,500 inheritance from a relative. She decided to invest the money in an account that earns 6% annual interest, compounded semiannually. The compound interest formula written as a function of time, t, is  $A(t) = P(1 + \frac{t}{n})^{nt}$ . A represents the amount of money in the account.

- P represents the initial deposit, or principal amount.
- r represents the annual interest rate as a decimal.
- *n* represents the number of compounding periods per year.
- *t* represents the number of years that the money has been invested.

Use the compound interest function and the information presented in the problem to write a function, A(t), that Angelica can use to determine her account balance after t years of investment. Simplify the expression in the function as completely as you can.

 $A(t) = 7,500(1 + \frac{0.06}{2})^{2t} = 7,500(1.03)^{2t}$ 

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**2.** Use your function to complete a table like the one shown. Round dollar values to the nearest cent.

TIME (YEARS)	0	0.5	1	1.5	2	2.5	3	
ACCOUNT BALANCE (DOLLARS)	7,500	7,725	7,956.75	8,195.45	8,441.32	8,694.56	8,955.39	

- Write an equation that Angelica can use to determine the length of time required for her account balance to be \$9,500.78.
   9,500.78 = 7,500(1.03)<sup>2t</sup>
- Extend your table or use graphing technology to generate a table. Use the table to approximate a solution to your equation.
   See margin.
- Use your function to write an equation Angelica could solve to determine the amount of time required for her account to reach a balance of \$12,800.
   12,800 = 7,500(1.03)<sup>2t</sup>
- **6.** Use a graph to approximate a solution to the equation. Explain how you used your graph to determine your estimate. *See margin.*
- 7. Interpret the approximate solution within the context of the problem. How do the coordinates of the intersection point relate to Angelica's actual account balance? (*Hint: How often is interest calculated on Angelica's account? Is the function A*(*t*) *continuous or discrete? Is the situation modeled by A*(*t*) *continuous or discrete?*) *See margin.*
- 8. ELPS Strategy Work with a partner. Tell your partner how you can estimate a solution to an equation from either a graph or a table. Use connecting words from the word bank shown to indicate the sequencing of your solution steps. *See margin.*

CONNECTING WORDS FOR TIME OR SEQUENCE AFTER LAST UNTIL SINCE THEN BEFORE NEXT NOW THAT

REFLECT

Compare and contrast how you approximate a solution to an exponential equation from a graph or from a table. How are they alike? How are they different? See margin.

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**8.** Possible response: An equation relates a function to a known output value. Now that you know the output value, locate it in the graph or table. Next, locate the paired input value that generates the output value. If you can't find the exact output value, find one that is close to it. After locating the input value, you can use that as an approximate solution to the equation.

## SUPPORTING ENGLISH LANGUAGE LEARNERS

The English language has a variety of connecting words that can be used to connect ideas in spoken or written language. As students are explaining their processes and comparing procedures, encourage students to speak using a variety of connecting words, such as the ones provided in the bank with the ELPS Strategy. Students should use connecting words with increasing accuracy and ease as they acquire more language skills (ELPS 3C).

**REFLECT ANSWER:** See page 798.

4. When x = 4, y = 9500.78, so A(4) = 9,500.78.

Angelica will have an account balance of about \$9,500.78 in 4 years.



## Y1=9500.77561041

6.  $h \approx 9$  years Possible process: Graph both A(t) and y = 12,800on a graphing calculator. For some technologies, A(t) will become y = A(x). Use the calculator to determine the coordinates of the point of intersection to be approximately (9.04, 12,800). These coordinates mean that when t = 9.04, A(t) = 12,800, so the solution to the equation is approximately 9.04.



7. Possible response: At exactly 9 years, Angelica's account balance will be \$12,678.25. The interest on Angelica's account is only calculated semiannually, or every 6 months (0.5 year).At 9.5 years, the next compounding period, Angelica's account balance will be \$13,151.30. Her account will never be exactly \$12,800, because while the function A(t) is continuous, the account balance is discrete since it is only calculated at 6-month intervals.

#### **REFLECT ANSWERS:**

Both graphs and tables show the relationship between specific input and output values. A table is an organized list and a graph is a collection of ordered pairs. Approximating the solution to an equation involves starting with a given output value and then using the graph or table to identify the input value that is paired with the given output value. When an exact value for the input is not found, you must round the input value which will give you an approximate solution.

An exponential function is continuous over its domain. *If a situation being modeled* by an exponential function is discrete instead of continuous, then only certain pairs of input-output values will apply to the situation. For example, with Angelica's situation, the *interest on her account is only* calculated every 0.5 years. So, while the function A(t) is continuous, the only values of A(t) that accurately model Angelica's account balance are those generated by input values that are multiples of 0.5. When you approximate the solution to an equation related to A(t), you need to observe whether or not the input value is one in which the account balance will actually be calculated.

 How does a situation that is discrete instead of continuous affect how you approximate a solution to an equation? See margin.

# EXPLAIN

Exponential functions are used to model situations with growth and decay that occur through repeated multiplication. You can use these functions to write related equations, then use graphs and tables to approximate solutions to the equations for the value of the independent variable that generates a particular value of the dependent variable. For exponential functions, there is a general growth and decay formula that you can use if you know the rate of increase or decrease as a percent.





In this formula, you would use 1 + r as the base of the exponential function if the rate, r, is an increase. You would use 1 - r as the base of the exponential function if the rate, r, is a decrease.

#### APPROXIMATING SOLUTIONS TO EQUATIONS GRAPHICALLY

Herman purchased a new truck for \$18,250. According to a popular vehicle website, Herman's truck should depreciate by 9% of its value each year. Herman can use an exponential growth and decay formula to write a function, v(t), to describe the value of his truck t years after its purchase.

$$v(t) = 18,250(1 - 0.09)^{t} = 18,250(0.91)^{t}$$

Herman wants to know how long it will be before his truck is worth half of what he paid for it. To solve this problem, use v(t) to write an equation. The value of the dependent variable, v(t), will be \$18,250  $\div$  2 = \$9,125.

$$9,125 = 18,250(0.91)^t$$

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Use graphing technology to graph v(t) and the line y = 9125 to determine the point on the graph of v(t) with a function value of 9125.

The point (7.35, 9125) is the intersection point of v(t) and y = 9125. Notice that 7.35 is a rounded value for t. Also notice that depreciation is a continuous event in that the value of Herman's truck changes throughout the year and not just at the end of the year.

t)  $\frac{1}{9}$   $\frac$ 

18,250 (0.91

We can approximate that v(7.35) = 9125 and estimate that when t = 7.35 years, the value of the truck will be \$9,125.



#### APPROXIMATING SOLUTIONS TO EQUATIONS TABULARLY

Herman also owns a valuable collection of baseball cards. Each year, the value of his collection increases by 8.5%. Currently, an appraiser said that the value of Herman's baseball card collection is \$2,750. Herman can use an exponential growth and decay formula to write a function, c(t), to describe the value of his collection t years from now.

$$c(t) = 2,750(1+0.085)^{t} = 2,750(1.085)^{t}$$

How long will it be before Herman's collection doubles in value? To solve this problem, write an equation from c(t). Before you do, you need to know that \$2,750, doubled in value, is \$5,500. The equation that is related to c(t) for this problem uses c(t) = 5,500.

#### $5,500 = 2,750(1.085)^t$

Use graphing technology to make a table of values for c(t). Look in the column for the dependent variable for a value of 5,500. You may need to refine the interval for  $\Delta x$  to see additional rational number values.

The table shown has one function value that is very close to \$5,500. When t = 8.5, c(t) = 5501.56. The target collection value, \$5,500, is slightly less than the identified value of c(t), so the *t*-value that generates \$5,500 will be slightly less than 8.5

Herman's baseball card collection will double in value in approximately 8.5 years.

TIME (YEARS), t	VALUE (DOLLARS), $c(t)$
8.0	5281.66
8.1	5324.93
8.2	5368.54
8.3	5412.52
8.4	5456.86
8.5	5501.56
8.6	5546.62

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#### INTEGRATING TECHNOLOGY

You can use a graphing calculator or app to estimate the solution of an exponential equation graphically. Graph the function in Y1 and then graph Y2 = n, where *n* is the given output value or function value. Use the calculator's or app's features to determine where the two graphs intersect. Usually, for an exponential function, the *x*-coordinate, which represents the solution to the equation, is a rounded value. Hence, the *x*-coordinate of the point of intersection is an approximation for the solution to the equation.

#### INTEGRATING TECHNOLOGY

You can use a graphing calculator or app to solve an equation tabularly. Enter the function into *Y*1 of the function editor. In the table feature, scroll up and down until you see the given output value or function value. If you do not see the exact function value, then change the *x*-interval to a smaller number (e.g., 0.1 instead of 1), and look again. The *x*-value in the same row as the function value is the solution to the equation.

#### APPROXIMATING SOLUTIONS TO EQUATIONS RELATED TO EXPONENTIAL FUNCTIONS

An exponential function has a base, which is a constant multiplier, and a power that contains the variable. An equation that is related to a given function, f(x), is one in which the value of the dependent variable is known and you need to determine the value(s) of the independent variable that generates it. For an exponential function, there will only be one pair of values for which this is true.

- Graphically, locate a point on the graph of f(x) that has a *y*-coordinate equal to the given function value. The *x*-coordinate of this point is the *x*-value paired with that function value. This *x*-value is the solution to the equation.
- Tabularly, locate the function value in the dependent variable column or row. The value in the independent variable column or row associated with this function value is the solution to the equation.

# EXAMPLE 1

Historical records of animal species extinctions have been kept since the 16th century. The graph shows the total number of documented extinctions that have taken place in 50-year time spans since the year 1600.

A researcher notes that the historical trend of an increase of about 62.8% per 50-year period can best be approximated as  $h(x) = 8(1.628)^x$ , where *x* represents the number of 50 year periods since



the year 1600 and h(x) represents the total number of documented extinctions. If the trend continues, in what year does the model predict there are 350 documented species extinctions? Write an equation related to h(x) and approximate the solution graphically.



## **ADDITIONAL EXAMPLE**

Janice buys a large fish tank and two guppies. The fish store warns her that the guppy population in her tank will increase by 135% every month. The function  $g(m) = 2(2.35)^m$ , where *m* is the number of months, represents the guppy population. Janice plans to sell the guppies when she has about 1000 in her tank. How long will it take for the guppy population to reach 1000? Write an equation, and use graphing technology to approximate the solution.

The equation whose solution yields an answer to the question is  $1000 = 2(2.35)^m$ . Using graphing technology, *m* is approximately 7.34 when g(m) = 1000. This means that Janice will have about 1000 guppies 7 months and 10 days after she purchases the initial two guppies.

## QUESTIONING STRATEGY

To help students recognize the meaning of various exponential functions, ask:

- Why does the function in Example 1 not follow the same pattern for writing the functions for Herman's truck and collection in the Explain section?
- How is the 50-year period represented in Example 1's function?



August of 1987.

# YOU TRY IT #1

A patient is admitted to the hospital for treatment of metastasized thyroid cancer. The prescribed treatment is to swallow a pill that contains 11 units of iodine-131, a radioactive isotope with a decay rate of 8.3% per day. In accordance with United States regulatory guidelines, the patient cannot be released from the hospital until no more than 1.2 units of iodine-131 remain. How long will the patient stay in the hospital? Write an equation, and approximate the solution graphically. *See margin.* 



#### YOU TRY IT! #1 ANSWER:

The model  $d(t) = 11(0.917)^t$ can be used to write the equation whose solution yields an answer to the question is  $1.2 = 11(0.917)^t$ . Solving this equation predicts that there will be 1.2 units of iodine-131 remaining after 25.57 days. Therefore, the patient can leave the hospital after approximately 26 days.



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## EXAMPLE 2

Before World War II, Sri Lanka had a public health crisis due to a large population of mosquitos that caused malaria outbreaks, increasing human death rates. After 1945, insect repellents were widely used that controlled the mosquito population, and the human death rate in Sri Lanka dropped by about 8.4% annually. The function  $p(x) = 22(0.916)^x$  represents the death rate in number of deaths per 1,000 people in the population of Sri Lanka, where *x* represents the time since 1945 in years.

TIME SINCE 1945, <i>x</i> (YEARS)	0	1	2	3	4	5	6	7	8	9
DEATH RATE, <i>p</i> ( <i>x</i> ) (DEATHS PER 1,000 PEOPLE)	22	20.2	18.5	16.9	15.5	14.2	13	11.9	10.9	10

The death rate in Sierra Leone, an African nation battling a recent outbreak of the Ebola virus, is approximately 11 people per 1,000. In what year did Sri Lanka have this same death rate? Write an equation related to p(x) that will answer the question, and use the table to approximate the solution.

**STEP 1** Examine the dependent value column of the table.

()	iINCE 1945, <i>x</i> (EARS)	0	1	2	3	4	5	6	7	8	9
DEATH (DE/ 1,000	DEATH RATE, <i>p</i> ( <i>x</i> ) (DEATHS PER 1,000 PEOPLE)		20.2	18.5	16.9	15.5	14.2	13	11.9	10.9	10
	A death rate of indicated dependent	11 pe nden	eople p t value	per th es of <i>p</i>	ousar $v(x)$ .	nd fall	ls bet	ween	these	e two	
STEP 2	Write an equa the question, approximately	Write an equation related to $p(x)$ whose solution answers the question, "In what year did Sri Lanka have a death rate of approximately 11 people per thousand?"									
	Since $p(x)$ represents the death rate per 1,000 people in Sri Lanka, write the equation $11 = 22(0.916)^{x}$ .										
STEP 3	Examine the independent value row of the table to determine the independent values of $p(x)$ that correspond to the dependent function values you indicated in Step 1.										
	dependent fu	nctio	n valu	ies yo	ou ine	dicate	ed in	Step	1.		
TIME S	SINCE 1945, <i>x</i> (EARS)	0	n valu	ies yo	ou ine 3	dicate	ed in 5	Step 6	1. 7	8	9
TIME S (1) DEATH (DEA 1,000	HATE, $p(x)$ ATHS PER D PEOPLE)	0 22	1 20.2	2 18.5	ou ine 3 16.9	4 15.5	ed in 5 14.2	Step 6 13	1. 7 11.9	8 10.9	9 10

#### **ADDITIONAL EXAMPLE**

Kris has an extensive comic book collection. He was hoping that the value of the comic books would increase each year; however, because his comic books are newer books, he has learned they are decreasing in value by 2% annually. His initial investment in his collection was \$17,356. Unfortunately he does not discover the depreciating value for a few years when he tries to sell his collection to purchase a new car. If the new car is \$15,500, how long after purchasing his collection does he need to sell it to have enough money to pay for the car? The function  $v(x) = 17,356(0.98)^x$ , where *x* represents the number of years, represents the value of Kris' collection. Write an equation related to v(x) that will answer the question, and use the table to approximate the solution.

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The equation whose solution yields an answer to the question is  $15,500 = 17,356(0.98)^x$ . Using the table, the collection would have a value of \$15,500 between 5 and 6 years after Kris purchased the collection. Using graphing technology to investigate further, the comic collection is \$15,500 approximately in 5.6 years. This means that Kris will need to sell his collection prior to about 5 years and 7 months to have enough money from the sale to pay for the new car.

YEARS, x	VALUE, $v(x)$
0	17,356.00
1	17.008.88
2	16,668.70
3	16,335.33
4	16,008.62
5	15,688.45
6	15,374.68

# **STEP 4** Use graphing technology with tables to determine the independent value of the function p(x) that corresponds most closely with a dependent value of 11.

Using graphing technology, you can change the interval between successive independent values in the table to tenths to determine that the independent value 7.9 corresponds to a dependent function value of 11. Therefore, p(7.9) = 11.

#### **STEP 5** Interpret the intersection point in terms of the situation.

A value of x = 7.9 means that the model shows that Sri Lanka had a death rate of 11 people per 1,000 in approximately the year 1945 + 7.9 = 1952.9. Convert 0.9 years to months: 0.9(12) = 10.8 or approximately 11 months.

The equation  $11 = 22(0.916)^x$  will yield the year in which Sri Lanka had a death rate of approximately 11 people per thousand. The model p(x) shows that Sri Lanka had a death rate of 11 people per thousand in November 1952.

TIME SINCE BEGINNING OF SENIOR YEAR IN HIGH SCHOOL, x (MONTHS)

0

1

2

3

4

5

6

7

8

9

VALUE OF COLLEGE SAVINGS, v(x)(DOLLARS)

\$23.250.00

\$23,366.25

\$23,483.08

\$23,600.50

\$23,718.50

\$23,837.09

\$23,956.28

\$24,076,06

\$24,196.44

\$24,317.42



When Martin begins his senior year of high school, he predicts the value of his college savings in an annuity that earns 6% annual interest compounded monthly. Martin uses the function  $v(x) = 23,250(1.005)^x$ , where  $6\% \div 12$  months is 0.005, *x* represents the number of months since his senior year in high school began, and v(x) represents the value of his college savings to create a table.

Martin will need \$24,060 to pay for his first year of college. How many months into his senior year of high school will his savings

completely pay for his first year of college? Write an equation related to v(x) that will answer the question, and use the table to approximate the solution.

See margin.

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#### YOU TRY IT! #2 ANSWER:

*The equation whose solution* answers the question is  $24060 = 23250(1.005)^{x}$ . The *model* v(x) = 23,250(1.005)*x* predicts that there will be \$24,060 in Martin's college savings after 6.866 months. *However, the interest earnings* on the account are only calculated once per month instead of continuously during the month. Therefore, it will take seven months into his senior year for the balance of Martin's college savings to completely pay for his first year of college.

TIME SINCE BEGINNING OF SENIOR YEAR IN HIGH SCHOOL, x (MONTHS)	VALUE OF COLLEGE SAVINGS, $v(x)$ (DOLLARS)
0	\$23,250.00
1	\$23,366.25
2	\$23,483.08
3	\$23,600.50
4	\$23,718.50
5	\$23,837.09
6	\$23,956.28
7	\$24,076,06
8	\$24,196.44
9	\$24,317.42

- **2.** At the end of 2030, the world population will be at 9.83 billion. By the end of 2031, the world population will be at 10.02 billion.
- 7. At the end of 20 years, Jake's account balance will be \$2984.68. At the end of 21 years, Jake's account balance will be \$3089.15.

PRACTICE /HOMEWORK

*Use the scenario below to complete questions* 1-5*.* 

## DEMOGRAPHY

The world population, f(x), is growing at a rate of 2% per year. In 2015, the world population was 7.3 billion. The function below represents this situation if *x* represents the number of years since 2015.

#### $f(x) = 7.3(1.02)^x$

- Write an equation that can be used to determine when the world's population will reach 10 billion.
   10 = 7.3(1.02)\*
- 2. The graph represents f(x), the world population, and y = 10. What do the points A and B represent on the graph?
  See margin.
- When will the world population be at exactly 10 billion?
   During the year 2031

The table shows the world population, f(x), based on the numbers of years since 2015.

- When will the world population reach 7.8 billion?
   During the year 2019
- Using the table in problem 4, when will the world population reach exactly 8 billion?
   During the year 2020

#### NUMBER OF YEARS WORLD POPULATION **SINCE 2015** (BILLIONS) f(x)0 7.3 1 7.446 2 7.595 3 7.747 4 7.902 5 8.060

(16.10.0

3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 2

(15, 9.83)

*Use the scenario below to complete questions* 6 – 10. **FINANCE** 

Jake invests \$1500 into an account that pays 3.5% interest compounded annually. The function f(x) represents the balance in his account after x years.

#### $f(x) = 1500(1 + 0.035)^x$

- **6.** Write an equation that can be used to determine when Jake's money will double.
  - 3000 = 1500(1.035)×

\$

**7.** The graph represents *f*(*x*) and *y* = 3000. What do the points *A* and *B* represent in the situation?See margin.

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#### 8. When will Jake's account balance reach exactly \$3,000? During the 21st year of his investment

The table shows some values of f(x), Jake's account balance based on x, the numbers of years the money has been invested.

- **9.** When will Jake's account balance reach \$1800? **During the 6th year of his investment**
- **10.** When will Jake's account balance reach \$1650? **During the 3rd year of his investment**

*Use the scenario below to complete questions* 11 - 15*.* 

#### FINANCE

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Oliver purchased a luxury sedan for \$32,450. After doing some research he discovered the vehicle will depreciate by 6% each year. The function f(x) represents the value of the vehicle after x years since it was purchased.

NUMBER OF YEARS x	ACCOUNT BALANCE (DOLLARS) f(x)
0	1500
1	1552.50
2	1606.84
3	1663.08
4	1721.28
5	1781.53
6	1843.88
7	1908.42

 $f(\mathbf{x}) = 32,450(0.94)^x$ 

- Write an equation that can be used to determine when the vehicle will be worth half of what Oliver paid for it.
   16,225 = 32,450(0.94)<sup>x</sup>
- 12. The graph represents f(x) and y = 16,225. What does point A represent in the situation?
  After 11.2 years, the vehicle will be worth \$16,225.



The table represents the value of the vehicle at the end of x years after purchase.

NUMBER OF YEARS, x	1	2	3	4	5	6	7	
VALUE OF VEHICLE	30503.00	28672 82	26952 /5	25355 30	23815 10	22286.27	210/310	
(DOLLARS), $f(x)$	50505.00	20072.02	20552.45	25555.50	25015.15	22500.27	21045.10	

- During which year will the vehicle first be worth less than \$30,000?
   During year 2
- During which year will the vehicle first be worth less than \$25,000?
   During year 5
- During which year will the vehicle be worth exactly \$22,000?
   During year 7

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#### **19.** *The deer population will* double in almost 5 years (4.959).



#### **20.** *The deer population will exceed* 2,000 during the 14th year.

x	f(x)
13	1845.84
14	2122.71
15	2441.12

#### WILDLIFE MANAGEMENT

A herd of deer contains 300 deer. The population of the herd increases at a rate of 15% per year assuming that there are no deaths among the herd. The function f(x) represents the deer population after *x* years. The function below represents this situation.

$$f(x) = 300(1.15)$$

- **16.** Use graphing technology to plot the function. See margin.
- **17.** Use a graphing calculator and the intersection feature to determine when the deer population will reach 425. See margin.
- **18.** Use a graphing calculator and the intersection feature to determine when the deer population will reach 500. See margin.
- **19.** Use a graphing calculator and the intersection feature to determine when the deer population will double. See margin.
- **20.** Use a graphing calculator and the table feature, determine during which year the deer population will exceed 2,000. See margin.