6.6 Solving Systems of Three Linear Equations

TEKS

AR.5E Represent and solve systems of three linear equations arising from mathematical and real-world situations using matrices and technology.

MATHEMATICAL PROCESS SPOTLIGHT

AR.1C Select tools, including real objects, manipulatives, paper and pencil, and technology as appropriate, and techniques, including mental math, estimation, and number sense as appropriate, to solve problems.

ELPS

4F Use visual and contextual support and support from peers and teachers to read grade-appropriate content area text, enhance and confirm understanding, and develop vocabulary, grasp of language structures, and background knowledge needed to comprehend increasingly challenging language.

VOCABULARY

matrix, system of equations, inverse matrix

MATERIALS

- graphing technology

FOCUSING QUESTION How can I use matrices and technology to represent and solve a system of three linear equations?

LEARNING OUTCOMES

- I can represent a problem with a system of three linear equations using matrices and technology.
- I can use matrices and technology to solve a problem involving a system of three linear equations.
- I can select tools, including paper and pencil and technology, as appropriate to solve

ENGAGE

Lashondra owns a Christmas tree farm. She wants to plant both Douglas fir trees and Scotch pine trees to sell next Christmas. Douglas fir trees cost \$250 per acre to plant and Scotch pine trees cost \$175 per acre to plant. Lashondra will plant 50 acres of trees and will spend \$11,000. How many acres of each tree will Lashondra plant? Define your variables, write a system of equations, and represent the system using a matrix equation.

See margin.

EXPLORE

Image source: Pixabay

Gulf Stream Lumber has a plant with three sawmills, *A*, *B*, and *C*. If all three sawmills run all day, then the three sawmills produce 5,700 board-feet of lumber. If sawmill *A* runs for two days and sawmill *B* runs for one day, they produce 4,900 board-feet of lumber. If only sawmills *B* and *C* run all day, then the two sawmills together produce 4,200 board-feet of lumber.

1. Write a system of equations that you could use to represent this problem. Let the variables *a*, *b*, and *c* represent the amount of lumber produced in one day by their respective sawmills.
See margin.

ENGAGE ANSWER;

Possible answer:

Let d represent the number of acres of Douglas fir tree and s represent the number of acres of Scotch pine trees.

$d + s = 50$

$250d + 175s = 11{,}000$

$$\begin{bmatrix} 1 & 1 \\ 250 & 175 \end{bmatrix}\begin{bmatrix} d \\ s \end{bmatrix} = \begin{bmatrix} 50 \\ 11{,}000 \end{bmatrix}$$

1. $a + b + c = 5{,}700$

$2a + b = 4{,}900$

$b + c = 4{,}200$

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2. Use the coefficients of a, b, and c to write matrix A where each row represents one equation and each column represents one of the variables a, b, and c. If an equation does not contain all three variables, be sure to use a term with 0 as a coefficient as a placeholder.
See margin.

3. Use the constants from each equation to write matrix B where each row represents one equation.
See margin.

4. Use the matrix $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to represent the variables and write a matrix equation relating matrix A, matrix B, and the variable matrix.
See margin.

> The inverse of a 3 × 3 matrix can be calculated using the formula shown.
>
> $$\text{If } A = \begin{bmatrix} a & b & c \\ d & f & g \\ h & j & k \end{bmatrix}, \text{ then } A^{-1} = \frac{1}{a(fk-gj)-b(dk-gh)-c(dj-fh)} \begin{bmatrix} fk-gj & cj-bk & bg-cf \\ gh-dk & ak-ch & cd-ag \\ dj-fh & bh-aj & af-bd \end{bmatrix}.$$

5. Use the inverse formula with paper and pencil, graphing technology (e.g., calculator or app), or an online matrix inverse calculator to calculate the matrix A^{-1}.
See margin.

6. Use the matrix A^{-1} and the matrix equation from a previous question to solve for $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.
See margin.

7. Write the solution to the system and explain what it means in the context of the original problem.
See margin.

> Repetivive computations such as those involved with calculating the inverse of a 3 × 3 matrix are easily programmed into a computer. Coding algorithms such as calculating the inverse of a matrix have many applications for computer programming.

Use the situation below to answer questions 8 – 10.

A candy company packages chocolate gift boxes using white chocolate, milk chocolate, and dark chocolate.

- A box of 1 package of white chocolate, 3 packages of milk chocolate, and 3 packages of dark chocolate weighs 14 ounces.
- A box of 1 package of white chocolate, 3 packages of milk chocolate, and 4 packages of dark chocolate weighs 17 ounces.
- A box of 1 package of white chocolate, 4 packages of milk chocolate, and 3 packages of dark chocolate weighs 15 ounces.

8. Let x represent the weight of one package of white chocolate, y represent the weight of one package of milk chocolate, and z represent the weight of one package of dark chocolate. Write a system of three linear equations that you can use to represent this problem.
See margin.

2. $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 5700 \\ 4900 \\ 4200 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 5700 \\ 4900 \\ 4200 \end{bmatrix}$

5. $A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \\ 2 & -1 & -1 \end{bmatrix}$

6. $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = [A]^{-1}[B]$

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} 5700 \\ 4900 \\ 4200 \end{bmatrix}$

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1500 \\ 1900 \\ 2300 \end{bmatrix}$

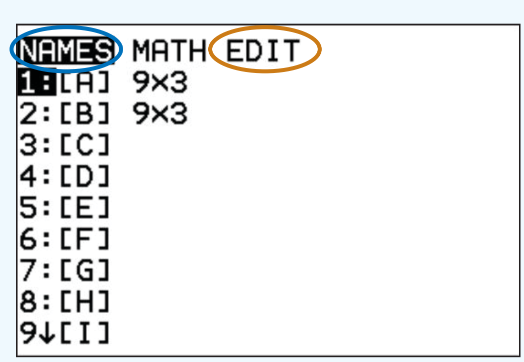
7. $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1500 \\ 1900 \\ 2300 \end{bmatrix}$, *so a = 1,500, b = 1,900, and c = 2,300. Sawmill A produces 1,500 board-feet of lumber in one day, sawmill B produces 1,900 board-feet of lumber in one day, and sawmill C produces 2,300 board-feet of lumber in one day.*

8. $x + 3y + 3z = 14$
$x + 3y + 4z = 17$
$x + 4y + 3z = 15$

INTEGRATING TECHNOLOGY

Use graphing calculators to multiply two matrices together. Depending on the technology, matrices can be created through the **edit** menu and can be used on the home screen for matrix operations using the **names** menu. Matrix operations can be performed on the calculator's home screen using the matrix names. To multiply $[A]^{-1}$ by $[B]$, use the Matrix-Names list to call up $[A]$, use the inverse (x^{-1}) key, the multiplication symbol, and then the Matrix-Names list to call up $[B]$.

$$[A]^{-1} \times [B]$$



9. Use technology to represent the system of three linear equations.
See margin.

10. Use technology to solve the system of three linear equations. Interpret the solution within the context of the problem.
See margin.

REFLECT

- **When solving a system of three linear equations using matrices, is it easier to use paper and pencil or technology to solve the problem? Explain your reasoning.**
  See margin.
- **How is the idea of left-multiplying by the inverse matrix to solve a matrix equation related to multiplying by the inverse to solve an equation with real numbers (e.g., to solve $2x = 7$, multiply by the inverse of 2)?**
  See margin.

EXPLAIN

Situations with three unknowns require particular pieces of information in order to solve for the values of those unknowns. To write a system of equations for the situation, you need to have as many equations as you do unknowns in order to solve the system. If the equations are all linear equations, then you can use matrices to solve the system of three linear equations.

Watch Explain and You Try It Videos

or click here

ELPS ACTIVITY

Read the material in this section with a partner. Use support from your peer by asking each other the following questions.

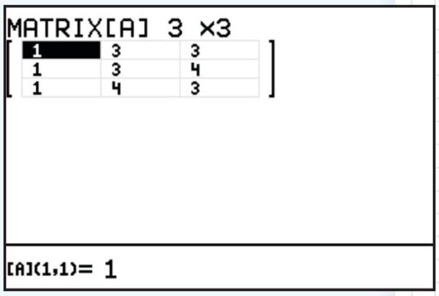
- Could you please summarize what this paragraph is saying?
- What word that you know is similar to this word?
- Where have you seen something like this before?

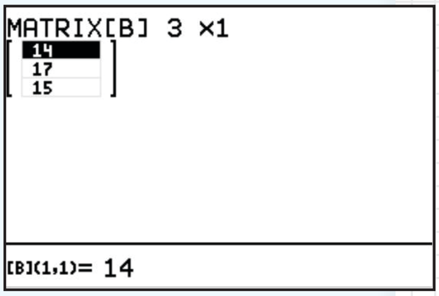
REPRESENTING A SYSTEM OF THREE LINEAR EQUATIONS USING MATRICES

For a system of three linear equations with three unknowns, you can use a matrix equation with 3 × 3 matrices to represent the system. This process is similar to what you did with systems of two linear equations with two unknowns.

- Make sure that all linear equations are in standard form, $Ax + By + Cz = D$. If one variable is missing from the equation, be sure to use a term with 0 as the coefficient as a placeholder.

9.





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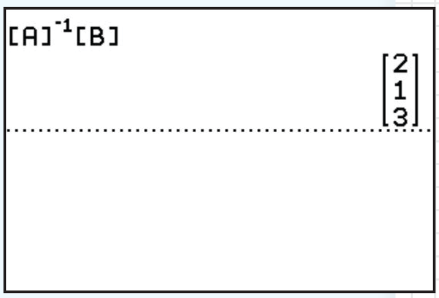


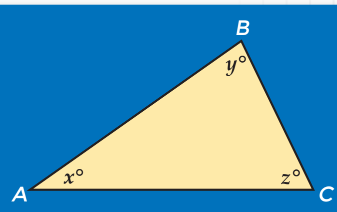
One package of white chocolate weighs 2 ounces, one package of milk chocolate weighs 1 ounce, and one package of dark chocolate weighs 3 ounces.

REFLECT ANSWERS:

Answers may vary. Possible response: For systems of three linear equations, it is much easier to use technology because to solve the system with matrices you need to know the inverse of the coefficient matrix. Calculating the inverse of a 3 × 3 matrix by hand using the formula is very tedious, and technology calculates this inverse much more quickly and efficiently.

Multiplying by the inverse matrix is a way of making the coefficient to the variable matrix equal to the identity matrix. Multiplying an equation by the inverse of the coefficient of x also makes the coefficient to the variable equal to the identity, or 1. Either way, you are using the inverse matrix/number to isolate the variable in order to solve the equation.

- Place the coefficients of the unknowns into a 3 × 3 coefficient matrix.
- Place the unknowns into a 3 × 1 variable matrix.
- Place the constants (D when the equation is in standard form) into a 3 × 1 constant matrix.



In ΔABC, the sum of the measures of the interior angles is 180°. Also, $m\angle B$ is twice $m\angle A$ and the sum of $m\angle A$ and $m\angle B$ is twice $m\angle C$. What are the measures of each interior angle of ΔABC?

Consider the triangle problem shown. If x, y, and z each represent the measure of one interior angle of ΔABC, then you can write the system of three linear equations shown.

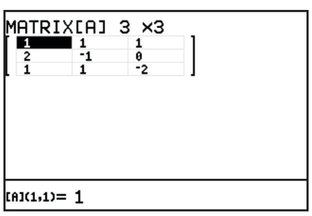
$$\begin{cases} x + y + z = 180 \\ y = 2x \\ x + y = 2z \end{cases} \rightarrow \begin{cases} x + y + z = 180 \\ 2x - y + 0z = 0 \\ x + y - 2z = 0 \end{cases} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 180 \\ 0 \\ 0 \end{bmatrix}$$

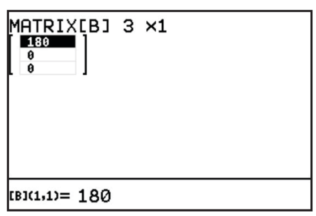
coefficient matrix

variable matrix

constant matrix

Once you have the system of three linear equations represented in a matrix, you can use technology to represent the system. In a graphing calculator, you enter the numeric values of each matrix entry into a matrix app.





DETERMINING THE INVERSE OF A 3 × 3 MATRIX

As you have seen with systems of two linear equations, in order to solve the matrix equation for the variable matrix, you need to left-multiply by the inverse of the coefficient matrix. In this case, the coefficient matrix is a 3 × 3 matrix.

For a 3 × 3 matrix, $A = \begin{bmatrix} a & b & c \\ d & f & g \\ h & j & k \end{bmatrix}$, the product of the inverse of matrix A, which is written as A^{-1}, and matrix A must be equal to the identity matrix.

$$A^{-1} \times \begin{bmatrix} a & b & c \\ d & f & g \\ h & j & k \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Solving this matrix equation for A^{-1} generates a formula for determining A^{-1} from a given matrix A.

If $A = \begin{bmatrix} a & b & c \\ d & f & g \\ h & j & k \end{bmatrix}$, then $A^{-1} = \frac{1}{a(fk-gj)-b(dk-gh)-c(dj-fh)} \begin{bmatrix} fk-gj & cj-bk & bg-cf \\ gh-dk & ak-ch & cd-ag \\ dj-fh & bh-aj & af-bd \end{bmatrix}$.

You can certainly calculate the inverse of a 3 × 3 matrix using paper and pencil and a calculator by hand. Or, you can use graphing technology, an app, or an online matrix inverse calculator to calculate the inverse of the coefficient matrix.

SOLVING A SYSTEM OF THREE LINEAR EQUATIONS USING MATRICES

Once you have written your matrix equation, you can use the inverse of the coefficient matrix to solve for the variable matrix. Remember that matrix multiplication is not commutative. So if you want the product of $[A]^{-1}$ and $[A]$ to equal the identity matrix, then you will need to left-multiply $[A]^{-1}$ by $[A]$.

$$[A]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [B] \longrightarrow [A]^{-1}[A]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [A]^{-1}[B]$$

With the triangle problem, you represented the situation with a system of three linear equations and then used the equations to create a matrix equation. You also used technology to represent the system of equations through matrices. Now, you can calculate the inverse of the coefficient matrix and begin multiplication.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 180 \\ 0 \\ 0 \end{bmatrix}$$

The inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix}$ can be calculated using either the inverse formula or technology.

$$[A]^{-1} = \begin{bmatrix} \frac{2}{9} & \frac{1}{3} & \frac{1}{9} \\ \frac{4}{9} & -\frac{1}{3} & \frac{2}{9} \\ \frac{1}{3} & 0 & -\frac{1}{3} \end{bmatrix}$$

Use $[A]^{-1}$ to solve the original matrix equation for the variable matrix, $\begin{bmatrix} x \\ y \\ Z \end{bmatrix}$.

$$[A]^{-1}[A]\begin{bmatrix} x \\ y \\ Z \end{bmatrix} = [A]^{-1}[B]$$

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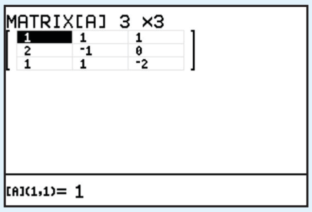
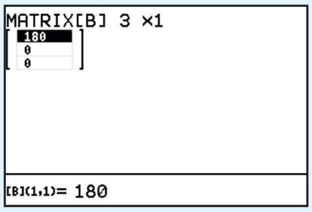
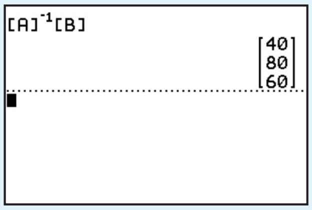
$$\begin{bmatrix} \frac{2}{9} & \frac{1}{3} & \frac{1}{9} \\ \frac{4}{9} & -\frac{1}{3} & \frac{2}{9} \\ \frac{1}{3} & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{9} & \frac{1}{3} & \frac{1}{9} \\ \frac{4}{9} & -\frac{1}{3} & \frac{2}{9} \\ \frac{1}{3} & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 180 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 40 \\ 80 \\ 60 \end{bmatrix}$$

The solution to the given system is (40, 80, 60). In the context of the problem, the measures of the three interior angles of ΔABC are 40°, 80°, and 60°.

USING TECHNOLOGY TO SOLVE SYSTEMS OF THREE LINEAR EQUATIONS WITH MATRICES

Graphing technology, such as a graphing calculator or app, can be extremely beneficial when solving systems of equations with matrices.

Write the system of linear equations as a matrix equation. Make sure that all three linear equations are in standard form.	$\begin{cases} x+y+z=180 \\ y=2x \\ x+y=2z \end{cases} \rightarrow \begin{cases} x+y+z=180 \\ 2x-y+0z=0 \\ x+y-2z=0 \end{cases}$ $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 180 \\ 0 \\ 0 \end{bmatrix}$
Enter the coefficient matrix into one matrix of your graphing technology.	
Enter the constant matrix into a second matrix of your graphing technology.	
Use matrix operations to calculate $[A]^{-1}[B]$.	

SYSTEMS OF THREE LINEAR EQUATIONS WITH MATRICES

Matrices can be used to represent and solve systems of three linear equations.

- Make sure that all three linear equations are in standard form, $Ax + By + Cz = D$, where A, B, C, and D are integers and at least one of A, B, and C is not equal to 0.
- Use the matrix equation $[A]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [B]$ where $[A]$ represents the coefficient matrix and $[B]$ represents the constant matrix.
- Determine the inverse of the coefficient matrix, $[A]^{-1}$.
- Left-multiply both sides of the matrix equation by $[A]^{-1}$. The left member of the equation, $[A]^{-1}[A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, is the identity matrix.
- Technology can be used to enter and calculate the values of the variable matrix, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [A]^{-1}[B]$.

EXAMPLE 1

Points X and Y are between points W and Z. The distance between points W and Z is thirty-six millimeters. The distance between points W and X is half the distance from point X to point Y. The distance between points Y and Z is one millimeter less than the distance from point X to point Y. What is the length in millimeters of each segment? Write a system of three linear equations to represent the situation and a matrix equation to represent the system. Represent and solve the system using matrices with technology.

STEP 1 **Define variables to represent the unknowns and use them write a system of linear equations to represent the system.**

Since the problem asks you to determine the length of the segments, the variables will represent the lengths, in millimeters, of the three segments. Let x represent the length of $\overline{WX}$, let y represent the length of $\overline{XY}$, and let z represent the length of $\overline{YZ}$.

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"The segment shown is a total of thirty-six millimeters long." → $x + y + z = 36$.
"The distance between points W and X is half the distance from point X to point Y." → $x = \frac{1}{2}y$.
"The distance between points Y and Z is one millimeter less than the distance from point X to point Y." → $z = y - 1$.

Therefore, a system of linear equations that represents the situation is

$$\begin{cases} x + y + z = 36 \\ x = \frac{1}{2}y \\ z = y - 1 \end{cases}$$

STEP 2 **Write a matrix equation that corresponds to the system of linear equations you wrote in Step 1. If necessary, rewrite any linear equations that are not already in standard form before writing the matrix equation.**

Two of the linear equations in the system you wrote in Step 1 are not in standard form, so it is necessary to rewrite both equations. Remember that if an equation does not contain all three variables, you should use a term with 0 as a coefficient as a placeholder.

$$\begin{cases} x + y + z = 36 \\ x = \frac{1}{2}y \\ z = y - 1 \end{cases} \rightarrow \begin{cases} x + y + z = 36 \\ x - \frac{1}{2}y + 0z = 0 \\ 0x - y + z = -1 \end{cases}$$

Although some of the variables appear to have no coefficients, remember that a coefficient of one is implied. A matrix equation that represents the system above is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 36 \\ 0 \\ -1 \end{bmatrix}.$$

coefficient matrix, variable matrix, constant matrix

STEP 3 **Represent and solve the system using matrices with technology.**

To represent the system using technology, enter a 3 × 3 matrix A with entries equal to the coefficient matrix and a 3 × 1 matrix B with entries equal to the constant matrix.

To solve the system using technology, multiply the inverse of matrix A by matrix B.

$$A^{-1}B = \begin{bmatrix} 7.4 \\ 14.8 \\ 13.8 \end{bmatrix}$$

ADDITIONAL EXAMPLE

In ΔDEF, the sum of the interior angles is 180°. $m\angle F$ is 6 times $m\angle E$. $m\angle D$ is 82 more than the sum of $m\angle E$ and $m\angle F$. What is the measure of each angle of ΔDEF? Write a system of three linear equations to represent the situation and a matrix equation to represent the system. Represent and solve the system using matrices with technology.

Let d represent m∠D, e represent m∠E, and f represent m∠F.

$$\begin{cases} d + e + f = 180 \\ f = 6e \\ d = e - f + 82 \end{cases} \rightarrow \begin{cases} d + e + f = 180 \\ 0d - 6e + f = 0 \\ d - e - f = 82 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -6 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 180 \\ 0 \\ 82 \end{bmatrix} \qquad A^{-1}B = \begin{bmatrix} 131 \\ 7 \\ 42 \end{bmatrix}$$

The measures of the angles of the triangle are m∠D=131°, m∠E=7°, and m∠F=42°.

$$\begin{cases} x + y + z = 36 \\ x = \frac{1}{2}y \\ z = y - 1 \end{cases}$$ *is the system of linear equations that represents this situation if x is the length in millimeters of $\overline{WX}$, y is the length in millimeters of $\overline{XY}$, and z is the length in millimeters of $\overline{YZ}$.*

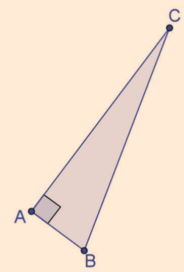
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 36 \\ 0 \\ -1 \end{bmatrix}$$ *is the matrix equation that represents the system of equations.*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7.4 \\ 14.8 \\ 13.8 \end{bmatrix}$$ *is the solution for the system determined using matrices and technology.*

The length of $\overline{WX}$ is 7.4 mm, the length of $\overline{XY}$ is 14.8 mm, and the length of $\overline{YZ}$ is 13.8 mm.

YOU TRY IT! #1

The perimeter of the right triangle shown is six hundred fifty centimeters. The length of its hypotenuse is one centimeter more than the length of its longer leg. The length of the longer leg of the right triangle is sixty-three centimeters less than fifteen times the length of its shorter leg. How long is each side of the right triangle? Write a system of three linear equations to represent the situation and a matrix equation to represent the system. Represent and solve the system using matrices with technology.



See margin.

EXAMPLE 2

A baseball stadium has three levels of seats: field level, mezzanine level, and upper level. Field level seat tickets sell for $29 each. Mezzanine level seat tickets sell for $19.99 apiece. Upper level seat tickets sell for $13.50. There are as many field level seats as there are mezzanine and upper level seats combined. The stadium has 58,000 seats and takes in $1,336,340 for every sold out game. How many seats are in each level of the baseball stadium? Write a system of three linear equations to represent the situation and a matrix equation to represent the system. Represent and solve the system using matrices with technology.

YOU TRY IT! #1 ANSWER:

If a is the length of the hypotenuse, b is the length of the longer leg, and c is the length of the shorter leg of the right triangle, then a system of linear equations that represents the situation is shown.

$$\begin{cases} a + b + c = 650 \\ a = b + 1 \\ b = 15c - 63 \end{cases} \rightarrow \begin{cases} a + b + c = 650 \\ a - b + 0c = 1 \\ 0a + b - 15c = -63 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -15 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 650 \\ 1 \\ -63 \end{bmatrix}$$

$$A^{-1}B = \begin{bmatrix} 313 \\ 312 \\ 25 \end{bmatrix}$$

The lengths of the legs of the right triangle are 25 centimeters and 312 centimeters and the length of its hypotenuse is 313 centimeters.

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STEP 1 **Define variables to represent the unknowns and use them write a system of linear equations to represent the system.**

Since the problem asks you to determine how many seats are in each level of the baseball stadium, the variables will represent the number of seats in each level. Let f represent the number of seats in the field level, let m represent the number of seats in the mezzanine level, and let u represent the number of seats in the upper level of the baseball stadium.

"There are as many field level seats as there are mezzanine and upper level seats combined." → $f = m + u$.
"The stadium has 58,000 seats...." → $f + m + u = 58{,}000$.
"Field level seat tickets sell for \$29 each. Mezzanine level seat tickets sell for \$19.99 apiece. Upper level seat ticket prices are \$13.50. …and takes in \$1,336,340 for every sold out game" → $29f + 19.99m + 13.50u = 1{,}336{,}340$.

Therefore, a system of linear equations that represents the situation is

$$\begin{cases} f = m + u \\ f + m + u = 58{,}000 \\ 29f + 19.99m + 13.50u = 1{,}336{,}340 \end{cases}$$

Step 2 **Write a matrix equation that corresponds to the system of linear equations you wrote in Step 1. If necessary, rewrite any linear equations that are not already in standard form before writing the matrix equation.**

One of the linear equations in the system you wrote in Step 1 is not in standard form, so it is necessary to rewrite that equation.

$$\begin{cases} f = m + u \\ f + m + u = 58{,}000 \\ 29f + 19.99m + 13.50u = 1{,}336{,}340 \end{cases} \rightarrow \begin{cases} f - m - u = 0 \\ f + m + u = 58{,}000 \\ 29f + 19.99m + 13.50u = 1{,}336{,}340 \end{cases}$$

Although some of the variables appear to have no coefficients, remember that a coefficient of one is implied. A matrix equation that represents the system above is

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 29 & 19.99 & 13.50 \end{bmatrix}\begin{bmatrix} f \\ m \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 58{,}000 \\ 1{,}336{,}340 \end{bmatrix}.$$

STEP 3 **Represent and solve the system using matrices with technology.**

To represent the system using technology, enter a 3×3 matrix A with entries equal to the coefficient matrix and a 3×1 matrix B with entries equal to the constant matrix.

ADDITIONAL EXAMPLE

A local school district is making a technology order. They need to upgrade the laptops and tablets on a campus. Some of the principals also need upgraded cell phones. Laptops cost \$715 each, tablets \$410, and cell phones \$245. They purchase 25 times more tablets than cell phones. The district spends a total of \$250,350 on 486 total devices.

Let l represent the number of laptops purchased, t represent the number of tablets purchased, and c represent the number of cell phones.

$$\begin{cases} l + t + c = 486 \\ t = 25c \\ 615l + 410t + 245c = 250350 \end{cases} \rightarrow \begin{cases} l + t + c = 486 \\ 0l + t - 25c = 0 \\ 715l + 410t + 245c = 250350 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -25 \\ 715 & 410 & 245 \end{bmatrix}\begin{bmatrix} l \\ t \\ c \end{bmatrix} = \begin{bmatrix} 486 \\ 0 \\ 250350 \end{bmatrix} \qquad A^{-1}B = \begin{bmatrix} 174 \\ 300 \\ 12 \end{bmatrix}$$

The district purchased 174 laptops, 300 tablets, and 12 cell phones.

To solve the system using technology, multiply the inverse of matrix A by matrix B.

$$A^{-1}B = \begin{bmatrix} 29{,}000 \\ 16{,}000 \\ 13{,}000 \end{bmatrix}$$

$$\begin{cases} f = m + u \\ f + m + u = 58{,}000 \\ 29f + 19.99m + 13.50u = 1{,}336{,}340 \end{cases}$$

is the system of linear equations that represents this situation where f is the number of seats in the field level, m is the number of seats in the mezzanine level, and u is the number of seats in the upper level of the baseball stadium.

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 29 & 19.99 & 13.50 \end{bmatrix}\begin{bmatrix} f \\ m \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 58{,}000 \\ 1{,}336{,}340 \end{bmatrix}$$

is the matrix equation that represents the system of equations.

$$\begin{bmatrix} f \\ m \\ u \end{bmatrix} = \begin{bmatrix} 29{,}000 \\ 16{,}000 \\ 13{,}000 \end{bmatrix}$$

is the solution for the system determined using matrices and technology.

The baseball stadium has 29,000 seats in its field level, 16,000 seats in its mezzanine level, and 13,000 seats in its upper level.

YOU TRY IT! #2

A snack company sells bulk nuts, fruits, and granola. The company plans to offer a new trail mix for $4.07 per pound that is composed of granola, peanuts, and raisins. There will be twice as many pounds of peanuts in the new trail mix as raisins. When sold separately, granola sells for $5.99 per pound, peanuts sell for $2.99 per pound, and raisins sell for $3.99 per pound. How many pounds of each ingredient are in 100 pounds of the new trail mix? Write a system of three linear equations to represent the situation and a matrix equation to represent the system. Represent and solve the system using matrices with technology.

See margin.

PRACTICE/HOMEWORK

For questions 1 – 3, create a matrix equation to represent each system of equations.

1. $\begin{cases} x + 2y - 3z = -2 \\ 2x - 2y + z = 7 \\ 2x + y + 3z = -4 \end{cases}$ **See margin.**

2. $\begin{cases} 20a + 9b = 127 \\ 8a + 18b + 3c = 97 \\ 3b + 5c = 14 \end{cases}$ **See margin.**

3. $\begin{cases} x + 2y + 3z = 9 \\ x = -2y \\ x + 4y - z = -5 \end{cases}$ **See margin.**

1.

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & -2 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \\ -4 \end{bmatrix}$$

2.

$$\begin{bmatrix} 20 & 9 & 0 \\ 8 & 18 & 3 \\ 0 & 3 & 5 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 127 \\ 97 \\ 14 \end{bmatrix}$$

3.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 0 \\ 1 & 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ -5 \end{bmatrix}$$

YOU TRY IT! #2 ANSWER:

Define the variables:

- *Let g be the amount in pounds of granola in 100 pounds of the new trail mix*
- *Let p be the amount in pounds of peanuts in 100 pounds of the new trail mix*
- *Let r be the amount in pounds of raisins in the 100 pounds of the new trail mix*

Write a system of linear equations, in standard form, that represents the situation.

$$\begin{cases} g + p + r = 100 \\ p = 2r \\ 5.99g + 2.99p + 3.99r = 4.07(100) \end{cases} \rightarrow \begin{cases} g + p + r = 100 \\ 0g + p - 2r = 0 \\ 5.99g + 2.99p + 3.99r = 407 \end{cases}$$

Write a matrix equation that represents this system.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 5.99 & 2.99 & 3.99 \end{bmatrix}\begin{bmatrix} g \\ p \\ r \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ 407 \end{bmatrix}$$

To represent the system using technology, enter a 3 × 3 matrix A with entries equal to the coefficient matrix and a 3 × 1 matrix B with entries equal to the constant matrix. To solve the system using technology, multiply the inverse of matrix A by matrix B

$$A^{-1}B = \begin{bmatrix} 28 \\ 48 \\ 24 \end{bmatrix}$$

There are 28 pounds of granola, 48 pounds of peanuts, and 24 pounds of raisins in 100 pounds of the new trail mix.

Use the situation described below to answer questions 4 – 8.

FINANCE

Carla wants to order gift bags of mixed dried fruits. Three of her options are described in the table below.

DRIED FRUIT BAGS

	DESCRIPTION	COST
MIXED BAG A	3 pounds each of pineapple and strawberries 2 pounds of apples	\$43.00
MIXED BAG B	4 pounds each of pineapple and apples 1 pound of strawberries	\$38.00
MIXED BAG C	5 pounds of pineapples 3 pounds of apples 2 pounds of strawberries	\$46.50

4. Write a system of equations that you could use to represent this situation. Let the variables x, y, and z represent the cost per pound of pineapple, apples, and strawberries, respectively.
See margin.

5. Use the coefficients of x, y, and z to write matrix A where each row represents one equation and each column represents one of the variables x, y, and z.
See margin.

6. Use the constants from each equation to write matrix B where each row represents one equation.
See margin.

7. Use the matrix $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to represent the variables and write a matrix equation relating matrix A, matrix B, and the variable matrix.
See margin.

8. Solve for $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and explain what each value means for the situation.
See margin.

Use the situation described below to answer questions 9 – 11.

BUSINESS

A distribution center is sending a shipment of 420 shoes worth a total of \$19,740 to a local store. The shipment contains three types of shoes: Shoe A has a value of \$50, Shoe B has a value of \$65, and Shoe C has a value of \$33.50. There are twice as many of shoe "C" than there are of shoe "B".

9. Write a system of equations that you could use to represent this situation. Let the variables a, b, and c represent the number of each type of shoe (A, B, and C) respectively.
See margin.

4. $\begin{cases} 3x + 2y + 3z = 43 \\ 4x + 4y + z = 38 \\ 5x + 3y + 2z = 46.5 \end{cases}$

5. Matrix A $\begin{bmatrix} 3 & 2 & 3 \\ 4 & 4 & 1 \\ 5 & 3 & 2 \end{bmatrix}$

6. Matrix B $\begin{bmatrix} 43 \\ 38 \\ 46.5 \end{bmatrix}$

7. $\begin{bmatrix} 3 & 2 & 3 \\ 4 & 4 & 1 \\ 5 & 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 43 \\ 38 \\ 46.5 \end{bmatrix}$

8. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3.5 \\ 8 \end{bmatrix}$

$x = 4$, so the dried pineapples cost \$4/lb.

$y = 3.5$, so the dried apples cost \$3.50/lb.

$z = 8$, so the dried strawberries cost \$8/lb.

9. $\begin{cases} 50a + 65b + 33.5c = 19{,}740 \\ a + b + c = 420 \\ c = 2b \end{cases}$

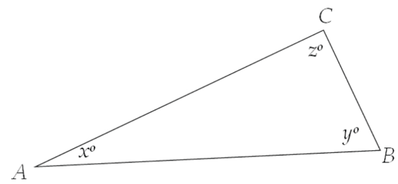
10. Create a matrix equation to represent this system of equations.
See margin.

11. Solve for $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, and explain what each value means for the situation.
See margin.

Use the situation described below to answer questions 12 – 14.

GEOMETRY

In the ΔABC, $m\angle C$ is three times $m\angle A$. Also, the sum of $m\angle A$ and $m\angle B$ is equal to $m\angle C$. Remember that in all triangles, the sum of the measures of the interior angles is 180°.



12. Write a system of equations that you could use to represent this situation.
See margin.

13. Create a matrix equation to represent this system of equations.
See margin.

14. Solve for $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and explain what each value means for the situation.
See margin.

Use the situation described below to answer questions 15 – 17.

FINANCE

Raul has an uncle who likes to play games when he visits. On his most recent visit, he told Raul that he could have all the money in his wallet if he were able to guess how many of each kind of bill he had. Here are the clues he gave:

- I have only 3 denominations of bills - \$5 bills, \$10 bills, and \$20 bills.
- I have a total of 17 bills in my wallet.
- The value of all the money in my wallet is \$180.
- I have twice as many \$5 bills as I do \$10 bills.

15. Write a system of equations that you could use to represent this situation. Let the variables *a*, *b*, and *c* represent the number of each type of bill (\$5, \$10, and \$20) respectively.
See margin.

16. Create a matrix equation to represent this system of equations.
See margin.

17. Solve for $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, and explain what each value means for the situation.
See margin.

10.

$$\begin{bmatrix} 50 & 65 & 33.5 \\ 1 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 19{,}740 \\ 420 \\ 0 \end{bmatrix}$$

11.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 210 \\ 70 \\ 140 \end{bmatrix}$$

a = 210, so there are 210 "A" shoes in the shipment

b = 70, so there are 70 "B" shoes in the shipment

c = 140, so there are 140 "C" shoes in the shipment

12. $\begin{cases} z = 3x \\ x + y = z \\ x + y + z = 180 \end{cases}$

13.

$$\begin{bmatrix} 3 & 0 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 180 \end{bmatrix}$$

14.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 30 \\ 60 \\ 90 \end{bmatrix}$$

x = 30, so $m\angle A$ is 30°
y = 60, so $m\angle B$ is 60°
z = 90, so $m\angle C$ is 90°

15. $\begin{cases} a + b + c = 17 \\ 5a + 10b + 20c = 180 \\ a = 2b \end{cases}$

16.

$$\begin{bmatrix} 1 & 1 & 1 \\ 5 & 10 & 20 \\ 1 & -2 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 17 \\ 180 \\ 0 \end{bmatrix}$$

17.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 5 \end{bmatrix}$$

a = 8, so there are 8 \$5 bills
b = 4, so there are 4 \$10 bills
c = 5, so there are 5 \$20 bills

720 CHAPTER 6: MATRICES

Use the situation described below to answer questions 18 – 20.

FINANCE

A school's Theater Club is performing a play for the general public. The table below shows the cost of each type of ticket the Theater Club will sell.

TICKET PRICES	
CHILD (UP TO 11 YEARS)	$2.50
STUDENT (12 – 18 YEARS)	$5.00
ADULT (19 YEARS AND UP)	$7.50

They sold 600 tickets and took in $3,085. The number of Student tickets sold was twice the sum of Child and Adult tickets.

18. Write a system of equations that you could use to represent the number of each type of ticket. Let the variables c, s, and a represent the number of each type of ticket (child, student, adult) respectively. Rewrite the system so that all equations are in standard form.
See margin.

19. Create a matrix equation to represent this system of equations.
See margin.

20. Solve the matrix equation, and explain what each value means for the situation.
See margin.

18. $$\begin{cases} c + s + a = 600 \\ 2.5c + 5s + 7.5a = 3085 \\ s = 2(c + a) \end{cases}$$

$$\begin{cases} c + s + a = 600 \\ 2.5c + 5s + 7.5a = 3085 \\ -2c + s - 2a = 0 \end{cases}$$

19.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2.5 & 5 & 7.5 \\ -2 & 1 & -2 \end{bmatrix}\begin{bmatrix} c \\ s \\ a \end{bmatrix} = \begin{bmatrix} 600 \\ 3085 \\ 0 \end{bmatrix}$$

20.

$$\begin{bmatrix} c \\ s \\ a \end{bmatrix} = \begin{bmatrix} 83 \\ 400 \\ 117 \end{bmatrix}$$

$c = 83$; they sold 83 child tickets

$s = 400$; they sold 400 student tickets

$a = 117$; they sold 117 adult tickets