Solving Systems of Two Linear Equations

6.5

FOCUSING QUESTION How can I use matrices to represent and solve a system of linear equations?

LEARNING OUTCOMES

- I can represent a problem with systems of linear equations using matrices.
- I can solve a problem with systems of linear equations using matrices.
- I can select tools, including paper and pencil and technology, as appropriate to solve problems.

ENGAGE

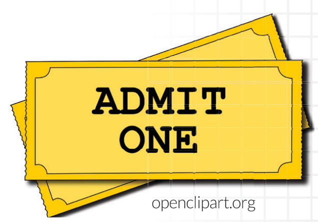
The Texas Cowboy Hall of Fame in Fort Worth, Texas, charges $5 for adult admission and $3 for children ages 5-12. A group of 12 people visits the museum and pays $56 for admission. Write a system of equations you could use to determine *c*, the number of children and *a*, the number of adults in the group.

$3c + 5a = 56$

$c + a = 12$

EXPLORE

The O'Neal High School drama club raised money for Project Prom by producing a play that was open to the community. On Friday night, they collected $1,344 from 64 student tickets and 96 adult tickets. On Saturday night, they collected $1,464 from 80 student tickets and 96 adult tickets. What is the cost of one student ticket and the cost of one adult ticket?



openclipart.org

1. Let x represent the cost of one student ticket and y represent the cost of one adult ticket. Write a system of equations that you could use to represent this problem.
   $64x + 96y = 1344$
   $80x + 96y = 1464$
2. Use the coefficients of x and y to write matrix A where each row represents one equation and each column represents one of the variables x or y.
   See margin.

TEKS

AR.5D Represent and solve systems of two linear equations arising from mathematical and real-world situations using matrices.

MATHEMATICAL PROCESS SPOTLIGHT

AR.1C Select tools, including real objects, manipulatives, paper and pencil, and technology as appropriate, and techniques, including mental math, estimation, and number sense as appropriate, to solve problems.

ELPS

4F Use visual and contextual support and support from peers and teachers to read grade-appropriate content area text, enhance and confirm understanding, and develop vocabulary, grasp of language structures, and background knowledge needed to comprehend increasingly challenging language.

VOCABULARY

matrix, system of equations, inverse matrix

MATERIALS

- graphing technology

2. $\begin{bmatrix} 64 & 96 \\ 80 & 96 \end{bmatrix}$

3. $\begin{bmatrix} 1344 \\ 1464 \end{bmatrix}$

4. $\begin{bmatrix} 64 & 96 \\ 80 & 96 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1344 \\ 1464 \end{bmatrix}$

9. *See margin below.*

10. $\begin{bmatrix} x \\ y \end{bmatrix} = [A]^{-1}[B]$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{16} & \frac{1}{16} \\ \frac{5}{96} & -\frac{1}{24} \end{bmatrix}\begin{bmatrix} 1344 \\ 1464 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7.5 \\ 9 \end{bmatrix}$

11-12. *See margin bottom of page 695.*

3. Use the constants from each equation to write matrix *B* where each row represents one equation.
See margin.

4. To represent a system of equations with matrices, use the matrix $\begin{bmatrix} x \\ y \end{bmatrix}$ to represent the variables. Write a matrix equation that multiplies the coefficient matrix *A* by the variable matrix to equal the constant matrix *B*.
See margin.

5. If you had an equation $Ax = B$, where *A* and *B* were real numbers, how would you solve for *x*?
Divide both sides of the equation by A, which is the same thing as multiplying by the inverse (reciprocal).

6. How are the operations of division and multiplication related?
Division is the inverse operation of multiplication.

7. To solve the matrix equation what should you multiply by in order to solve for *x*? Remember that matrix multiplication is not commutative, so be sure to specify whether the multiplication should be left-multiplication or right-multiplication.
Left-multiply by the inverse of A, or A^{-1}.

8. Write a matrix equation relating *A*, A^{-1}, $\begin{bmatrix} x \\ y \end{bmatrix}$, and *B*.
$[A]^{-1}[A]\begin{bmatrix} x \\ y \end{bmatrix} = [A]^{-1}[B]$, so $\begin{bmatrix} x \\ y \end{bmatrix} = [A]^{-1}[B]$

One way to indicate the **inverse** of a number, variable, function, or matrix is to use an exponent of −1. For example, the inverse of $f(x)$ is $f^{-1}(x)$ and the multiplicative inverse of 3 is $3^{-1} = \frac{1}{3}$.

The inverse of a 2 × 2 matrix can be calculated using the formula shown.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

9. Use the inverse formula to calculate the matrix A^{-1}.
See margin.

10. Use the matrix A^{-1} and the matrix *B* to solve the matrix equation from the previous question for $\begin{bmatrix} x \\ y \end{bmatrix}$. Simplify the product completely.
See margin.

11. Use graphing technology such as a graphing calculator or app to enter the data from matrix *A* and matrix *B*. Remember that matrices are defined by the number of rows by the number of columns.
See margin.

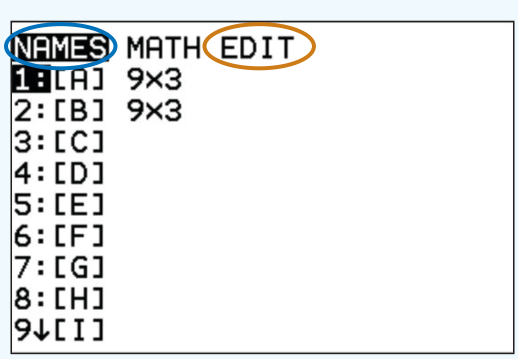
12. Use matrix operations to left-multiply the inverse of matrix *A* by matrix *B*, $[A]^{-1} \times [B]$. How does this matrix compare to matrix $\begin{bmatrix} x \\ y \end{bmatrix}$, the one you calculated in a previous question?
See margin.

9.
$$A^{-1} = \frac{1}{(64)(96)-(96)(80)}\begin{bmatrix} 96 & -96 \\ -80 & 64 \end{bmatrix} = -\frac{1}{-1536}\begin{bmatrix} 96 & -96 \\ -80 & 64 \end{bmatrix} = \begin{bmatrix} -\frac{1}{16} & \frac{1}{16} \\ \frac{5}{96} & -\frac{1}{24} \end{bmatrix}$$

INTEGRATING TECHNOLOGY

Use graphing calculators to multiply two matrices together. Depending on the technology, matrices can be created through the **edit** menu and can be used on the home screen for matrix operations using the **names** menu. Matrix operations can be performed on the calculator's home screen using the matrix names. To multiply $[A]^{-1}$ by $[B]$, use the Matrix-Names list to call up $[A]$, use the inverse (x^{-1}) key, the multiplication symbol, and then the Matrix-Names list to call up $[B]$.

$$[A]^{-1} \times [B]$$



694 CHAPTER 6: MATRICES

13. The matrix for $\begin{bmatrix} x \\ y \end{bmatrix}$ identifies the values of x and y in the system of equations. Write the solution to the system and explain what it means in the context of the original problem.
See margin.

REFLECT

- **The matrix equation $[A]\begin{bmatrix} x \\ y \end{bmatrix} = [B]$ represents the system of linear equations relating x and y. What operation do you need to do to this equation to solve for $\begin{bmatrix} x \\ y \end{bmatrix}$? Why?**
  See margin.
- **Why do you left-multiply by the inverse of matrix A instead of right-multiplying?**
  See margin.

EXPLAIN

Certain types of real-world situations can be represented using systems of equations. Systems of equations are used for situations with multiple unknowns when you are given certain pieces of information about how those unknowns are related.

Watch Explain and You Try It Videos

or click here

There are also several ways to solve systems of linear equations with two variables. For example, consider the problem shown.

> The perimeter of a rectangle is 8 feet. The length of the rectangle is 5 feet less than twice the width. What are the length and width of the rectangle?

In previous courses, you learned that systems of linear equations can be solved with graphs, tables, and algebraic methods including substitution and elimination.

ELPS CONNECTION

Study the graphic organizer shown. With a partner, use the visual graphic organizer along with the contextual support of the perimeter problem to study the different methods that could be used to solve the perimeter problem. How does the visual and contextual support enhance and confirm your understanding of systems of equations?

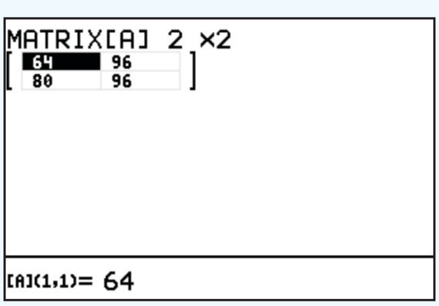
13. *See margin below.*

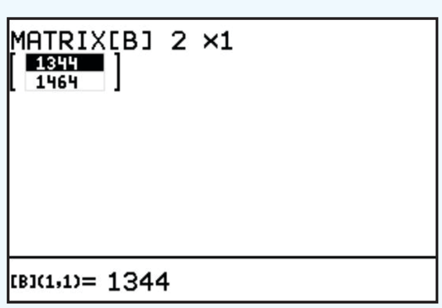
REFLECT ANSWERS:

To solve for $\begin{bmatrix} x \\ y \end{bmatrix}$, you need to un-multiply by [A]. With matrices, un-multiplication is done by multiplying by the inverse matrix in order to produce the identity matrix.

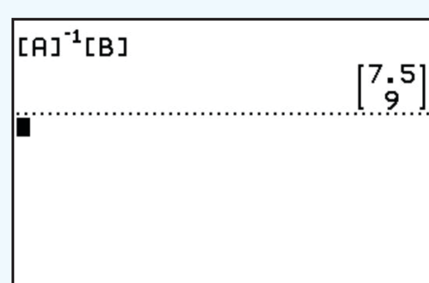
If you right-multiply by $[A]^{-1}$, then you will have $[A]\begin{bmatrix} x \\ y \end{bmatrix}[A]^{-1}$. Since matrix multiplication is not commutative, this will not make $[A] \times [A]^{-1} = [I]$. Left-multiplication, $[A]^{-1}[A]$, multiplies to produce the identity matrix.

11.





12. *The product of $[A]^{-1} \times [B]$ is the same as the matrix $\begin{bmatrix} x \\ y \end{bmatrix}$.*



13. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7.5 \\ 9 \end{bmatrix}$, *so $x = 7.5$ and $y = 9$. Since x represents the cost of one student ticket and y represents the cost of one adult ticket, one student ticket costs \$7.50, and one adult ticket costs \$9.00*

QUESTIONING STRATEGIES

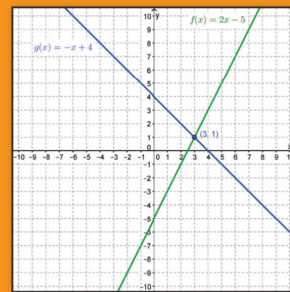
Guide students in considering why matrices might be a better option for solving systems of equations by asking:

- Looking at the four methods for solving systems, how did you decide which method to use?
- Considering the Explore problem, could using matrix multiplication be an easier method? Why or why not?

SOLVING SYSTEMS OF LINEAR EQUATIONS

GRAPHING

Graph both equations and look for the point of intersection.



TABLES

Make a table of values for both equations. Look for the x-value that generates the same y-value for both equations.

x	$Y1$	$Y2$
2	-1	2
3	1	1
4	3	0

SUBSTITUTION

Solve one equation for one variable in terms of the other. Substitute this expression into the second equation.

$$\begin{cases} y = 2x - 5 \\ 2x + 2y = 8 \end{cases}$$

$$\begin{aligned} 2x + 2(2x - 5) &= 8 \\ 2x + 4x - 10 &= 8 \\ 6x - 10 &= 8 \\ 6x &= 18 \\ x &= 3 \\ y = 2(3) - 5 &= 1 \end{aligned}$$

ELIMINATION OR ADDITION

Place both equations in standard form. Multiply one equation by an integer that will cause one variable to sum to 0 when added to the second equation.

$$\begin{cases} y = 2x - 5 \\ 2x + 2y = 8 \end{cases}$$

$2x - y = 5 \rightarrow 2x - y = 5$

$-(2x + 2y = 8)$ $\quad -2x - 2y = -8$

$-3y = -3$

$y = 1$

$$\begin{aligned} 2x + 2(1) &= 8 \\ 2x + 2 &= 8 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

There are other ways to solve systems of linear equations. One way is to use matrices.

DETERMINING THE INVERSE OF A MATRIX

When representing and solving systems of linear equations using matrices, you must use the inverse of a matrix. Technology can calculate the inverse of a matrix, but you can also perform these calculations by hand.

For a 2 × 2 matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the product of matrix A and the inverse of matrix A, which is written as A^{-1}, must be equal to the identity matrix.

$$A^{-1} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solving this matrix equation for A^{-1} generates a formula for determining A^{-1} from a given matrix A.

The product of a number and its multiplicative inverse is 1. For example, the multiplicative inverse of $\frac{2}{5}$ is $\frac{5}{2}$ because $\frac{2}{5} \times \frac{5}{2} = \frac{10}{10} = 1$. Likewise, the product of a matrix and its inverse is the identity matrix consisting only of 1's and 0's. For a 2 × 2 matrix, the **identity matrix** is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

696 CHAPTER 6: MATRICES

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

REPRESENTING A SYSTEM OF LINEAR EQUATIONS USING MATRICES

For a system of two linear equations with two unknowns, you can use a matrix equation with 2×2 matrices to represent the system. First, make sure that both linear equations are in standard form, $Ax + By = C$. Then, place the coefficients of the unknowns into a 2×2 coefficient matrix, the unknowns into a 2×1 variable matrix, and the constants (C when the equation is in standard form) into a 2×1 constant matrix.

Let's look back at the perimeter problem. If x represents the width of the rectangle and y represents the length of the rectangle, then you can write the system of two linear equations shown.

$$\begin{cases} y = 2x - 5 \\ x + y = 4 \end{cases} \longrightarrow \begin{cases} 2x - y = 5 \\ x + y = 4 \end{cases} \longrightarrow \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

coefficient matrix
variable matrix
constant matrix

SOLVING A SYSTEM OF LINEAR EQUATIONS USING MATRICES

Once you have written your matrix equation, you can use the inverse of the coefficient matrix to solve for the variable matrix. Remember that matrix multiplication is not commutative. When you use matrix multiplication to solve a matrix equation, there is a variable matrix. You will need to place the inverse matrix on the left side of the equation making left-multiplication necessary on both sides of the equation.

$$[A]\begin{bmatrix} x \\ y \end{bmatrix} = [B] \longrightarrow [A]^{-1}[A]\begin{bmatrix} x \\ y \end{bmatrix} = [A]^{-1}[B]$$

For example, if you have the matrix equation shown, you can calculate the inverse of the coefficient matrix and begin multiplication. In this equation, the original perimeter equation from the perimeter problem, $2x + 2y = 8$ has been simplified to $x + y = 4$.

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

The inverse of $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ can be calculated using the inverse formula.

$$[A]^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$[A]^{-1} = \frac{1}{2(1) - (-1)(1)}\begin{bmatrix} 1 & -(-1) \\ -(1) & 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Use $[A]^{-1}$ to solve the original matrix equation for the variable matrix, $\begin{bmatrix} x \\ y \end{bmatrix}$.

$$[A]^{-1}[A]\begin{bmatrix} x \\ y \end{bmatrix} = [A]^{-1}[B]$$

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3}(5) + \frac{1}{3}(4) \\ -\frac{1}{3}(5) + \frac{2}{3}(4) \end{bmatrix}$$

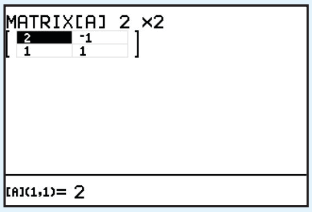
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3}(5) + \frac{1}{3}(4) \\ -\frac{1}{3}(5) + \frac{2}{3}(4) \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{9}{3} \\ \frac{3}{3} \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

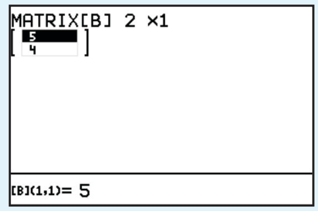
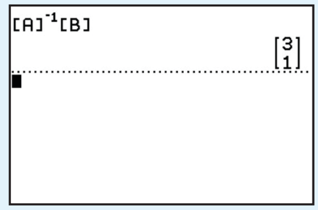
The solution to the given system is (3, 1). In the context of the problem, the rectangle has a width of 3 feet and a length of 1 foot.

USING TECHNOLOGY TO SOLVE SYSTEMS OF TWO LINEAR EQUATIONS WITH MATRICES

Graphing technology, such as a graphing calculator or app, can be extremely beneficial when solving systems of equations with matrices.

Write the system of linear equations as a matrix equation. Make sure that both linear equations are in standard form.	$\begin{cases} y = 2x - 5 \\ x + y = 4 \end{cases} \rightarrow \begin{cases} 2x - y = 5 \\ x + y = 4 \end{cases}$ $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$
Enter the coefficient matrix into one matrix of your graphing technology.	

698 CHAPTER 6: MATRICES

Enter the constant matrix into a second matrix of your graphing technology.	
Use matrix operations to calculate $[A]^{-1}[B]$.	

SYSTEMS OF TWO LINEAR EQUATIONS WITH MATRICES

Matrices can be used to represent and solve systems of two linear equations.

- Make sure that both linear equations are in standard form, $Ax + By = C$, where A, B, and C are integers, $A \neq 0$, and $B \neq 0$.
- Use the matrix equation $[A]\begin{bmatrix} x \\ y \end{bmatrix} = [B]$ where $[A]$ represents the coefficient matrix and $[B]$ represents the constant matrix.
- Determine the inverse of the coefficient matrix, $[A]^{-1}$.
- Left-multiply both sides of the matrix equation by $[A]^{-1}$. In the left member of the equation, $[A]^{-1}[A] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is the identity matrix.
- Technology can be used to enter and calculate the values of the variable matrix, $\begin{bmatrix} x \\ y \end{bmatrix} = [A]^{-1}[B]$.

EXAMPLE 1

Two angles are supplementary. The difference in their measures is thirty degrees. What are the measures of the two angles? Write a system of linear equations and its corresponding matrix equation to represent the situation. Solve the system using matrices.

STEP 1 Define the variables to represent the unknowns and use them to write a system of linear equations that represents the situation.

QUESTIONING STRATEGIES

Help students recall previously learned vocabulary:

- What relationship do supplementary angles share?
- What relationship do complementary angles share?

Since the problem asks you to determine the measures of the angles, the variables will represent the measures in degrees of the two angles. Let x represent the measure in degrees of the larger angle, and let y represent the measure in degrees of the smaller angle.

"The two angles are supplementary" → $x + y = 180$.
"The difference in their measures is thirty degrees." → $x - y = 30$.

Therefore, a system of linear equations that represents the situation is

$$\begin{cases} x + y = 180 \\ x - y = 30 \end{cases}$$

STEP 2 **Write a matrix equation that corresponds to the system of linear equations you wrote in Step 1. If necessary, rewrite any linear equations that are not already in standard form before writing the matrix equation.**

Both linear equations in the system you wrote in Step 1 are in standard form, so it is not necessary to rewrite either equation.

An understood coefficient of one is implied for both x and y.

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 180 \\ 30 \end{bmatrix}$$

STEP 3 **Solve the system using matrices by finding the inverse of the coefficient matrix and left-multiplying both sides of the matrix equation by it.**

Recall that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Since $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $A^{-1} = \frac{1}{(1)(-1)-(1)(1)}\begin{bmatrix} -1 & -(1) \\ -(1) & 1 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

Use the inverse of the coefficient matrix to solve the system.

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 180 \\ 30 \end{bmatrix}$$

$$\begin{bmatrix} \left(\frac{1}{2}\right)(1) + \left(\frac{1}{2}\right)(1) & \left(\frac{1}{2}\right)(1) + \left(\frac{1}{2}\right)(-1) \\ \left(\frac{1}{2}\right)(1) + \left(-\frac{1}{2}\right)(1) & \left(\frac{1}{2}\right)(1) + \left(-\frac{1}{2}\right)(-1) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}\right)(180) + \left(\frac{1}{2}\right)(30) \\ \left(\frac{1}{2}\right)(180) + \left(-\frac{1}{2}\right)(30) \end{bmatrix}$$

ADDITIONAL EXAMPLE

The sum of two angles is 110 degrees. The larger angle is equal to three times the smaller angle. What are the measures of the angles? Write a system of linear equations and its corresponding matrix equation to represent the system. Solve the system using matrices.

$$\begin{cases} x + y = 110 \\ y = 3x \end{cases} \rightarrow \begin{cases} x + y = 110 \\ 3x - y = 0 \end{cases}$$

$$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 110 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 27.5 \\ 82.5 \end{bmatrix}$$

The measure of the larger angle is 82.5 degrees, and the measure of the smaller angle is 27.5 degrees.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 90+15 \\ 90-15 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 105 \\ 75 \end{bmatrix}$$

$\begin{cases} x+y=180 \\ x-y=30 \end{cases}$ *is the system of linear equations that represents this situation if x is the measure in degrees of the larger angle and y is the measure in degrees of the smaller angle.*

$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 180 \\ 30 \end{bmatrix}$ *is the matrix equation that represents the system of equations*

$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 105 \\ 75 \end{bmatrix}$ *is the solution for the system determined using matrices.*

The measure of the larger angle is 105 degrees and the measure of the smaller angle is 75 degrees.

YOU TRY IT! #1

Two angles are complementary. Three times the measure of the larger angle is ten degrees less than four times the measure of the smaller angle. Write a system of linear equations and its corresponding matrix equation to represent the system. Solve the system using matrices.

See margin.

EXAMPLE 2

A high school's football team won its first game with the help of its kicker. He scored a total of nine points in field goals and extra points after touchdowns to help his team win by a narrow two-point margin. His tough week of practice paid off. He attempted to score five times in the game and kicked the football through the uprights each time. How many extra points and how many field goals did the high school football kicker kick in the game that he helped his team win? Write a system of linear equations and its corresponding matrix equation to represent the system. Solve the system using matrices.

STEP 1 Define variables to represent the unknowns and use them write a system of linear equations to represent the system.

YOU TRY IT! #1 ANSWER:

Let x represent the measure of the larger angle and y represent the measure of the smaller angle.

$$\begin{cases} x+y=90 \\ 3x=4y-10 \end{cases} \rightarrow \begin{cases} x+y=90 \\ 3x-4y=-10 \end{cases}$$

$$\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 90 \\ -10 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 50 \\ 40 \end{bmatrix}$$

The measure of the larger angle is 50 degrees, and the measure of the smaller angle is 40 degrees.

Let t represent the number of extra points and f represent the number of field goals that the kicker kicked to help his team win the game.

"He scored a total of nine points…". Since extra points after touchdowns are worth a single point and field goals are worth three points, then $t + 3f = 9$.
"He attempted to score five times…and kicked the football through the uprights each time." Therefore, $t + f = 5$.

A system of linear equations that represents the situation is

$$\begin{cases} t + 3f = 9 \\ t + f = 5 \end{cases}$$

STEP 2 **Write a matrix equation that corresponds to the system of linear equations you wrote in Step 1. If necessary, rewrite any linear equations that are not already in standard form before writing the matrix equation.**

Both linear equations in the system you wrote in Step 1 are in standard form, so it is not necessary to rewrite either equation.

$$\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} t \\ f \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$

STEP 3 **Solve the system using matrices by finding the inverse of the coefficient matrix and left-multiplying both sides of the matrix equation by it.**

Recall that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Since $A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$, $A^{-1} = \frac{1}{(1)(1)-(3)(1)}\begin{bmatrix} 1 & -(3) \\ -(1) & 1 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 1 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

Use the inverse of the coefficient matrix to solve the system.

$$\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} t \\ f \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} \left(-\frac{1}{2}\right)(1) + \left(\frac{3}{2}\right)(1) & \left(-\frac{1}{2}\right)(3) + \left(\frac{3}{2}\right)(1) \\ \left(\frac{1}{2}\right)(1) + \left(-\frac{1}{2}\right)(1) & \left(\frac{1}{2}\right)(3) + \left(-\frac{1}{2}\right)(1) \end{bmatrix}\begin{bmatrix} t \\ f \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}\right)(9) + \left(\frac{3}{2}\right)(5) \\ \left(\frac{1}{2}\right)(9) + \left(-\frac{1}{2}\right)(5) \end{bmatrix}$$

ADDITIONAL EXAMPLE

Winona asks her math teacher how many multiple choice and free response questions will be on the upcoming test over matrices. Sensing a teachable moment, her teacher gives her a riddle: there are 40 problems on the upcoming test, all multiple choice or free response. The multiple choice problems are 2.2 points each, and the free response problems count at 3 point a piece. Knowing the total points on the test to be 100, write a system of linear equations and its corresponding matrix equation to represent the system. Solve the system using matrices.

$$\begin{cases} m + f = 40 \\ 2.2m + 3f = 100 \end{cases}$$

$$\begin{bmatrix} 1 & 1 \\ 2.2 & 3 \end{bmatrix}\begin{bmatrix} m \\ f \end{bmatrix} = \begin{bmatrix} 40 \\ 100 \end{bmatrix}$$

$$\begin{bmatrix} m \\ f \end{bmatrix} = \begin{bmatrix} 25 \\ 15 \end{bmatrix}$$

There will be 25 multiple choice and 15 free response questions on Winona's upcoming math test.

$$\begin{bmatrix} \left(-\frac{1}{2}\right)+\left(\frac{3}{2}\right) & \left(-\frac{3}{2}\right)+\left(\frac{3}{2}\right) \\ \left(\frac{1}{2}\right)+\left(-\frac{1}{2}\right) & \left(\frac{3}{2}\right)+\left(-\frac{1}{2}\right) \end{bmatrix}\begin{bmatrix} t \\ f \end{bmatrix}=\begin{bmatrix} -\frac{9}{2}+\frac{15}{2} \\ \frac{9}{2}-\frac{5}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} t \\ f \end{bmatrix}=\begin{bmatrix} \frac{6}{2} \\ \frac{4}{2} \end{bmatrix}$$

$$\begin{bmatrix} t \\ f \end{bmatrix}=\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$\begin{cases} t+3f=9 \\ t+f=5 \end{cases}$ *is the system of linear equations that represents this situation if t is the number of extra points kicked after touchdowns and f is the number of field goals kicked.*

$\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} t \\ f \end{bmatrix}=\begin{bmatrix} 9 \\ 5 \end{bmatrix}$ *is the matrix equation that represents the system of equations.*

$\begin{bmatrix} t \\ f \end{bmatrix}=\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ *is the solution for the system determined using matrices.*

The high school football team's kicker made three successful extra point kicks after touchdowns in addition to two field goals.

YOU TRY IT! #2

A teacher prepares a side dish of peas and carrots to bring to a faculty potluck luncheon. The recipe calls for six cups of vegetables. The teacher spends $7.89 at a grocery store to purchase fresh carrots for $1.28 per cup and peas for $1.34 per cup. How many cups of peas and carrots are in the teacher's side dish? Write a system of linear equations and its corresponding matrix equation to represent the system. Solve the system using matrices.

See margin.

YOU TRY IT! #2 ANSWER:

Let p represent the number of cups of peas and let c represent the number of cups of carrots in the teacher's side dish.

$$\begin{cases} p+c=6 \\ 1.34p+1.28c=7.89 \end{cases}$$

$$\begin{bmatrix} 1 & 1 \\ 1.34 & 1.28 \end{bmatrix}\begin{bmatrix} p \\ c \end{bmatrix}=\begin{bmatrix} 6 \\ 7.89 \end{bmatrix}$$

$$\begin{bmatrix} p \\ c \end{bmatrix}=\begin{bmatrix} 3.5 \\ 2.5 \end{bmatrix}$$

The teacher put three and a half cups of peas and two and a half cups of carrots in her side dish for a faculty potluck luncheon.

EXAMPLE 3

Charlotte and Kaylee sing in their high school's choir. For this year's fundraiser, the choir director has chosen to sell scented bars of soap in addition to the usual plastic cups with the high school's logo on them. Charlotte sold 126 bars of soap and 64 plastic cups, raising a total of $1,350. Kaylee raised $1,340 by selling 43 bars of soap and 100 plastic cups. How much did the choir charge for bars of soap and plastic cups? Select pencil and paper or technology as appropriate to represent and solve this problem using matrices with a system of linear equations.

STEP 1 **Select paper and pencil or technology to define variables to represent the unknowns and use them write a system of linear equations to represent the system.**

You can choose to use paper and pencil to define variables and write a system of equations. Let x represent the price of a bar of soap and y represent the price of a plastic cup the choir members sold for this year's fundraiser.

The equation that represents Charlotte's sales is $126x + 64y = 1350$.
The equation that represents Kaylee's sales is $43x + 100y = 1340$.

A system of linear equations that represents the situation is

$$\begin{cases} 126x + 64y = 1350 \\ 43x + 100y = 1340 \end{cases}$$

STEP 2 **Write a matrix equation that corresponds to the system of linear equations you wrote in Step 1. If necessary, rewrite any linear equations that are not already in standard form before writing the matrix equation.**

Both linear equations in the system you wrote in Step 1 are in standard form, so it is not necessary to rewrite either equation.

$$\begin{bmatrix} 126 & 64 \\ 43 & 100 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1350 \\ 1340 \end{bmatrix}$$

STEP 3 **Solve the system using matrices.**

The values in the coefficient matrix and constant matrix are quite large. While you could use pencil and paper to find the inverse matrix and left-multiply it to both sides of the matrix equation, it would take a significant amount of time to do so. You can choose to use technology to find the inverse matrix and also to multiply it by the constant matrix.

ADDITIONAL EXAMPLE

Rayanne and Callie bake and deliver cookies and cakes to raise money for an upcoming band trip. Cookies are sold by the dozen, and cakes are sold individually. Rayanne sells 15 dozen cookies and 5 cakes earning her $312.50 for her trip. Callie sells 18 dozen cookies and 9 cakes earning her $450 toward the trip. How much did their customers pay for a dozen cookies or one cake? Write a system of linear equations and its corresponding matrix equation to represent the system. Solve the system using matrices.

$$\begin{cases} 15x + 5y = 312.50 \\ 18x + 9y = 450 \end{cases}$$

$$\begin{bmatrix} 15 & 5 \\ 18 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 312.50 \\ 450 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12.50 \\ 25 \end{bmatrix}$$

Rayanne and Callie sold cookies for $12.50 per dozen and cakes for $25 each.

The example below shows the problem worked out using pencil and paper. Refer to the part of this section entitled, **USING TECHNOLOGY TO SOLVE SYSTEMS OF TWO LINEAR EQUATIONS WITH MATRICES** to recall how technology may be used instead.

Recall that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Let $A = \begin{bmatrix} 126 & 64 \\ 43 & 100 \end{bmatrix}$.

$$A^{-1} = \frac{1}{(126)(100)-(64)(43)}\begin{bmatrix} 100 & -(64) \\ -(43) & 126 \end{bmatrix} = \frac{1}{9848}\begin{bmatrix} 100 & -64 \\ -43 & 126 \end{bmatrix}$$

Use the inverse of the coefficient matrix to solve the system.

$$\frac{1}{9848}\begin{bmatrix} 100 & -64 \\ -43 & 126 \end{bmatrix} \cdot \begin{bmatrix} 126 & 64 \\ 43 & 100 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{9848}\begin{bmatrix} 100 & -64 \\ -43 & 126 \end{bmatrix} \cdot \begin{bmatrix} 1350 \\ 1340 \end{bmatrix}$$

$$\frac{1}{9848}\begin{bmatrix} (100)(126) + (-64)(43) & (100)(64) + (-64)(100) \\ (-43)(126) + (126)(43) & (-43)(64) + (126)(100) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{9848}\begin{bmatrix} (100)(1350) + (-64)(1340) \\ (-43)(1350) + (126)(1340) \end{bmatrix}$$

$$\frac{1}{9848}\begin{bmatrix} 12600 + (-2752) & 6400 + (-6400) \\ (-5418) + 5418 & (-2752) + 12600 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{9848}\begin{bmatrix} 135000 + (-85760) \\ (-58050) + 168840 \end{bmatrix}$$

$$\frac{1}{9848}\begin{bmatrix} 9848 & 0 \\ 0 & 9848 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{9848}\begin{bmatrix} 49240 \\ 110790 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ \frac{45}{4} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 11.25 \end{bmatrix}$$

Charlotte and Kaylee's high school choir charged $5.00 for each scented bar of soap and $11.25 for each plastic cup with the high school's logo printed on it for this year's fundraiser.

YOU TRY IT! #3

After a year, a teacher's investments in an individual retirement account (IRA) that pays 4% interest and a savings account that pays 1.5% interest have earned $125. The teacher's invested a total of $5,000. Select pencil and paper or technology as appropriate to represent and solve this problem using matrices with a system of linear equations.

See margin.

YOU TRY IT! #3 ANSWER:

Represent the system using pencil and paper but solve the system using matrices with technology. Let r represent the amount the teacher invested in the invidual retirement account (IRA) and s represent the amount the teacher invested in a savings account.

$$\begin{cases} 0.04r + 0.015s = 125 \\ r + s = 5000 \end{cases}$$

$$\begin{bmatrix} 0.04 & 0.015 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 125 \\ 5000 \end{bmatrix}$$

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 2000 \\ 3000 \end{bmatrix}$$

The teacher invested $2,000 in the individual retirement account and $3,000 in the savings account.

PRACTICE/HOMEWORK

For questions 1 – 8, write a matrix equation to represent the system of linear equations and solve the system using matrices.

1. $\begin{cases} x + y = 9 \\ x - y = 3 \end{cases}$ **See margin.**

2. $\begin{cases} 2x + 3y = 12 \\ 2x - y = 4 \end{cases}$ **See margin.**

3. $\begin{cases} 3x + 5y = -4 \\ 4x + 2y = 18 \end{cases}$ **See margin.**

4. $\begin{cases} 5x + 2y = -5 \\ 7x + 3y = -6 \end{cases}$ **See margin.**

5. $\begin{cases} x - 2y = 7 \\ 2x - 7y = 11 \end{cases}$ **See margin.**

6. $\begin{cases} 1.5x - 0.2y = 8.3 \\ 2.5x + 1.4y = 6.9 \end{cases}$ **See margin.**

7. $\begin{cases} \frac{1}{2}x - \frac{1}{4}y = 3 \\ \frac{3}{4}x + \frac{1}{2}y = 1 \end{cases}$ **See margin.**

8. $\begin{cases} 3x - 7y = -5 \\ x + 3y = 1 \end{cases}$ **See margin.**

Use the following scenario for questions 9 - 11.

FINANCE

Tickets for a soccer game are $6 for adults and $4 for students. At the last game there were 220 tickets sold and $1064 collected.

9. Write a system of linear equations to represent the situation.
a + s = 220
6a + 4s = 1064

10. Write a matrix equation that corresponds to the system of linear equations you wrote in question 9.
See margin.

11. Solve the system using matrices to determine how many adult tickets and student tickets were sold.
92 adult tickets and 128 student tickets

Use the following scenario for questions 12 - 14.

FINANCE

Allison has $5.10 in quarters and dimes in her piggy bank. She has 27 coins in all.

12. Write a system of linear equations to represent the situation.
q + d = 27
0.25q + 0.10d = 5.10

13. Write a matrix equation that corresponds to the system of linear equations you wrote in question 12.
See margin.

14. Solve the system using matrices to determine how many quarters and dimes Allison has in her piggy bank.
16 quarters and 11 dimes

1.
$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

2.
$$\begin{bmatrix} 2 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 4 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

3.
$$\begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 18 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \end{bmatrix}$$

4.
$$\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ -6 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

5.
$$\begin{bmatrix} 1 & -2 \\ 2 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$$

6.
$$\begin{bmatrix} 1.5 & -0.2 \\ 2.5 & 1.4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8.3 \\ 6.9 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

7.
$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{3}{4} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

8.
$$\begin{bmatrix} 3 & -7 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

10.
$$\begin{bmatrix} 1 & 1 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} a \\ s \end{bmatrix} = \begin{bmatrix} 220 \\ 1064 \end{bmatrix}$$

13.
$$\begin{bmatrix} 1 & 1 \\ 0.25 & 0.10 \end{bmatrix}\begin{bmatrix} q \\ d \end{bmatrix} = \begin{bmatrix} 27 \\ 5.10 \end{bmatrix}$$

Use the following scenario for questions 15 - 17.

CRITICAL THINKING

A test has 28 questions that total 100 points. The test contains multiple choice questions that are worth 3 points each and short answer questions that are worth 5 points each.

15. Write a system of linear equations to represent the situation.
m + s = 28
3m + 5s = 100

16. Write a matrix equation that corresponds to the system of linear equations you wrote in question 15.
See margin.

17. Solve the system using matrices to determine how many multiple choice questions and how many short answer questions are on the test.
20 multiple choice questions and 8 short answer questions

Use the following scenario for questions 18 - 20.

CRITICAL THINKING

Ryan is playing a number game with his little brother Andrew. Ryan said he was thinking of two numbers and wanted Andrew to figure out the two numbers. Ryan said that two times the smaller number plus three times the larger number is forty-five. Also, three times the smaller number plus two times the larger number is forty.

18. Write a system of linear equations to represent the situation.
2s + 3l = 45
3s + 2l = 40

19. Write a matrix equation that corresponds to the system of linear equations you wrote in question 18.
See margin.

20. Solve the system using matrices to determine what the two numbers are that Ryan is thinking about.
6 is the smaller number and 11 is larger number

16.
$$\begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} m \\ s \end{bmatrix} = \begin{bmatrix} 28 \\ 100 \end{bmatrix}$$

19.
$$\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} s \\ l \end{bmatrix} = \begin{bmatrix} 45 \\ 40 \end{bmatrix}$$