TEKS

AR.4D Determine the linear factors of a polynomial function of degree two and of degree three when represented symbolically and tabularly and graphically where appropriate.

MATHEMATICAL PROCESS SPOTLIGHT

AR.1C Select tools, including real objects, manipulatives, paper and pencil, and technology as appropriate, and techniques, including mental math, estimation, and number sense as appropriate, to solve problems.

ELPS

2I Demonstrate listening comprehension of increasingly complex spoken English by following directions, retelling or summarizing spoken messages, responding to questions and requests, collaborating with peers, and taking notes commensurate with content and grade-level needs.

VOCABULARY

zeroes, linear, factors, polynomial

MATERIALS

- graphing technology

5.8 Decomposing Polynomial Functions

FOCUSING QUESTION How can graphs, tables, and symbolic representations be used to determine linear factors of polynomial functions?

LEARNING OUTCOMES

- I can determine the linear factors of polynomial functions from tables, graphs, or symbolic representations.
- I can select tools, including paper and pencil or technology, to solve problems.

ENGAGE

Which of the following functions have a linear factor of $2x - 1$?

- $2x^2 - x$
- $2x^2 + 7x - 4$
- $2x^2 - x - 3$
- $4x^3 - 10x^2 + 4x$

$2x^2 - x$, $2x^2 + 7x - 4$, and $4x^3 - 10x^2 + 4x$

Joshua Tree National Park

EXPLORE

Work with a partner. For each of the following pairs of polynomial functions, each partner should select one function. Write the polynomial function as a product of linear factors. Take turns explaining to your partner how you determined the linear factors. Select one of the questions from the Question Bank to ask your partner, listening carefully to the answer. Check your factors using graphing technology.

QUESTION BANK

- ✔ How did you determine the linear factors?
- ✔ From which representation is it easier for you to determine the linear factors? Explain your reasoning.
- ✔ How does the degree of the polynomial compare to the number of linear factors you could have?

1.

x	-3	-2.5	-2	-1.5	-1	-0.5
$f(x)$	0	-2	-3	-3	-2	0

$f(x) = (x + 3)(2x + 1)$

x	-3	-2	-1	0	1	2
$g(x)$	-30	-8	0	0	-2	0

$g(x) = x(x + 1)(x - 2)$

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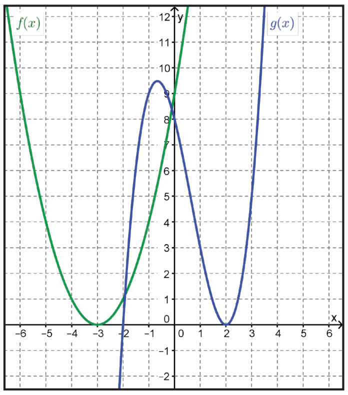
2.

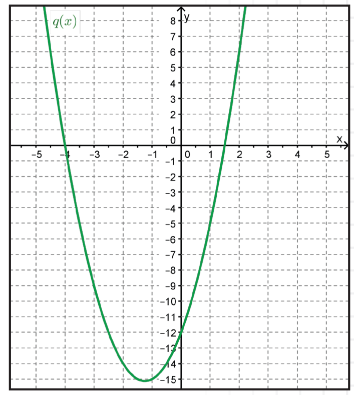


f(x) = (x + 3)²
g(x) = (x + 2)(x – 2)²

3. $f(x) = x^2 + 10x + 21$, $g(x) = x^3 + 5x^2 + 2x - 8$
f(x) = (x + 3)(x + 7)
g(x) = (x – 1)(x + 2)(x + 4)

4.

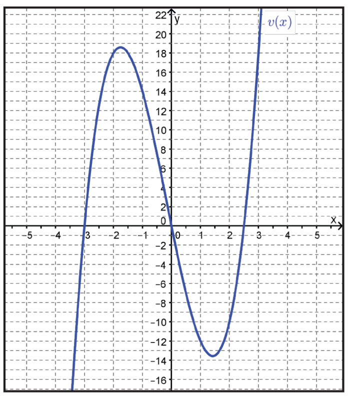
x	$p(x)$
-2	80
-1	21
$-\frac{1}{2}$	$2\frac{3}{4}$
$-\frac{2}{5}$	0
0	-8
$\frac{1}{2}$	$-11\frac{3}{4}$
1	-7
$1\frac{1}{5}$	$1\frac{4}{5}$
$1\frac{1}{3}$	0
2	24
$2\frac{3}{4}$	$66\frac{15}{16}$



q(x) = (x + 4)(2x – 3)

p(x) = (5x + 2)(3x – 4)

5. $t(x) = 4x^2 + 3x - 27$



t(x) = (4x – 9)(x + 3)
v(x) = x(x + 3)(2x – 5)

6.

x	-2	-1	0	1	2	3	4	5
$b(x)$	-50	-16	0	4	2	0	4	20

$c(x) = 2x^3 - 3x^2 - 14x$
b(x) = x(x – 3)²; c(x) = x(x + 2)(2x – 7)

REFLECT

- **What similarities are there among factoring polynomial functions from a table, graph, or symbolic representation?**
  See margin.
- **What differences are there among factoring polynomial functions from a table, graph, or symbolic representation?**
  See margin.

INTEGRATING TECHNOLOGY

Encourage students to use technology to graph functions given in symbolic form. With the graphing technology, students can calculate the zeroes of the functions and then use those zeroes to write linear factors.

For functions given in table form, students can use graphing technology to create a scatterplot or use finite differences to determine whether the function is quadratic (degree two) or cubic (degree three).

SUPPORTING ENGLISH LANGUAGE LEARNERS

As students are learning the English language, they need the opportunity to demonstrate their listening comprehension. One way is to pair students up and have them ask questions and respond to questions (**ELPS 2I**) that are commensurate with content and grade-level needs.

REFLECT ANSWERS:

Factoring from any representation relies on identifying zeroes of a function and then using those zeros to determine linear factors that will make the function value equal to zero for a given x-value.

Using a table relies on the x-coordinate of the x-intercept, using a graph relies on a zero of the function, and using symbolic representations relies on the numerical relationships among the terms. If there is a constant factor that vertically dilates the graph or reflects it across the x-axis, factoring by tables and graphs may not reveal that constant factor. However, factoring from a symbolic representation reveals that constant factor as a common factor among all terms in the function.

EXPLAIN

Watch Explain and You Try It Videos

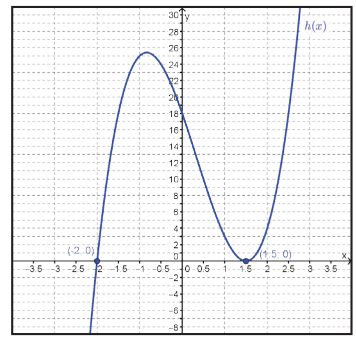
or click here

The zeroes of a polynomial function are related to the function's linear factors. You can see these relationships when the function is represented in a table, graph, or with symbols.

Look at the graph of $h(x) = 4x^3 - 4x^2 - 15x + 18$. There are two x-intercepts in the graph: $(-2, 0)$ and $(1.5, 0)$. Each x-intercept corresponds to a zero of $h(x)$. A cubic function has up to 3 unique zeroes. Notice how the graph of $h(x)$ touches, but does not cross, the x-axis at $(1.5, 0)$. That means the linear factor represented by $(1.5, 0)$ is repeated, so the linear factor will be used twice.

A table of values for $h(x)$ also reveals the zeroes of $h(x)$. The x-value corresponding with $h(x) = 0$ is the zero of $h(x)$.

A zero of $h(x)$ is the x-coordinate of an x-intercept of $h(x)$. From the graph and the table, you can identify two zeros of $h(x)$ that can be used to determine the linear factors of $h(x)$.



x	$h(x)$
-3	-81
$-2\frac{1}{2}$	-32
-2	0
$-1\frac{1}{2}$	18
-1	25
$-\frac{1}{2}$	24
0	18
$\frac{1}{2}$	10
1	3
$1\frac{1}{2}$	0
2	4

$(-2, 0) \rightarrow$ zero: -2
$(1.5, 0) \rightarrow$ zero: $1.5 = 1\frac{1}{2} = \frac{3}{2}$

When a zero is of the form $\frac{a}{b}$, the linear factor is $(bx - a)$.

zero: $-2 \rightarrow 1x - (-2) = x + 2$
zero: $\frac{3}{2} \rightarrow 2x - 3$

The function $h(x)$ can be rewritten in factored form, $h(x) = (x + 2)(2x - 3)(2x - 3)$ or $h(x) = (x + 2)(2x - 3)^2$

General Form **Factored Form**

$h(x) = 4x^3 - 4x^2 - 15x + 18 \leftrightarrow h(x) = (x + 2)(2x - 3)^2$

Polynomial Form

DIFFERENTIATING INSTRUCTION

Marian Small and Amy Lin (*More Good Questions: Great Ways to Differentiate Secondary Mathematics Instruction*, 2010) recommend using open questions as a way to provide multiple entry points for students to interact with the mathematics they are learning in a way that honors what they know. In this section, use an open question such as:

Compare the zeroes of these three functions. What do you notice?

$f(x) = 4x^2 - 17x + 4$ $\quad g(x) = 2x^2 - 5x + 2$ $\quad h(x) = 9x^2 - 82x + 9$

Add another function that acts the same way.

Solution: The zeroes of each function are reciprocals of each other. f(x) has zeroes of 4 and $\frac{1}{4}$, g(x) has zeroes of 2 and $\frac{1}{2}$, and h(x) has zeroes of 9 and $\frac{1}{9}$. Another function with this pattern is $j(x) = 5x^2 - 26x + 5$.

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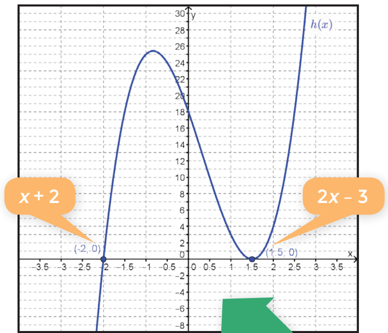
So for $h(x)$, the linear factors can be identified from graphs, tables, or symbolic representations.



x	$h(x)$
-3	-81
$-2\frac{1}{2}$	-32
-2	0
$-1\frac{1}{2}$	18
-1	25
$-\frac{1}{2}$	24
0	18
$\frac{1}{2}$	10
1	3
$1\frac{1}{2}$	0
2	4

x + 2

2x – 3

GRAPH

TABLE

$$h(x) = 4x^3 - 4x^2 - 15x + 18$$

SYMBOLIC

$$h(x) = (x + 2)(2x - 3)^2$$

DECOMPOSING POLYNOMIAL FUNCTIONS

Decomposing a polynomial function means to write it as a product of factors. Zeroes of the function can be used to identify linear factors of the polynomial function.

- In a graph, zeroes of a polynomial function can be identified from the x-coordinates of the x-intercepts.
- In a table, zeroes of a polynomial function can be identified by an x-value that generates a function value of 0.
- In a symbolic representation, some polynomial functions can be factored using algebraic methods, including:
  - ✓ Trial and Error
  - ✓ Perfect Square Trinomials
  - ✓ Difference of Squares or Cubes
  - ✓ Sum of Cubes
  - ✓ Perfect Cube Polynomials (four terms)
  - ✓ Grouping

ADDITIONAL EXAMPLES

Given the tables of values below, write the quadratic or cubic functions as a product of linear factors. Also, using the properties of algebra, write it as a polynomial function.

1.

x	$f(x)$
-6	9
-5.5	4
-5	0
-4.5	-3
-4	-5
-3.5	-6
-3	-6
-2.5	-5
-2	-3
-1.5	0
-1	4

$f(x) = (x + 5)(2x + 3) = 2x^2 + 13x + 15$

2.

x	$g(x)$
0	16
1	9
2	4
3	1
4	0
5	1
6	4
7	9
8	16

$g(x) = (x - 4)^2 = x^2 - 8x + 16$

3.

x	$h(x)$
-4	-18
-3.5	-5.5
-3	0
-2.5	0
-2	-4
-1.5	-10.5
-1	-18
-0.5	-25
0	-20
0.5	-31.5
1	-28
1.5	-18
2	0

$h(x) = (x + 3)(2x + 5)(x - 2) = 2x^3 + 7x^2 - 7x - 30$

EXAMPLE 1

Given the table of values, write a cubic function, $f(x)$, as a product of linear factors. Graph the function to verify.

x	-3	-2	-1	0	1	2	3	4	5	6
$f(x)$	-24	0	6	0	-12	-24	-30	-24	0	48

STEP 1 The table of values shows three zeroes for $f(x)$. The function values are zero when x is -2, 0, and 5. Using that information, write the three factors of $f(x)$.

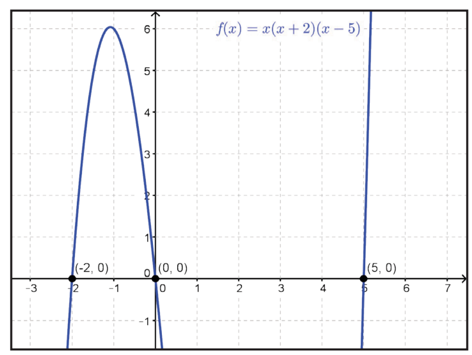
$x = -2$ $x = 0$ $x = 5$

$x + 2 = 0$ $x - 5 = 0$

STEP 2 Write the function $f(x)$ in factored form.

$f(x) = x(x + 2)(x - 5)$

STEP 3 Graph the function. Verify that the zeroes in the graph are the same as x-coordinates for the function values of zero in the table.



YOU TRY IT! #1

Given the table of values, write the quadratic function $g(x)$ as a product of linear factors. Also, using the properties of algebra, write it as a polynomial function.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$g(x)$	7	0	-5	-8	-9	-8	-5	0	7

$g(x) = (2x + 3)(2x - 3)$ or $g(x) = 4x^2 - 9$

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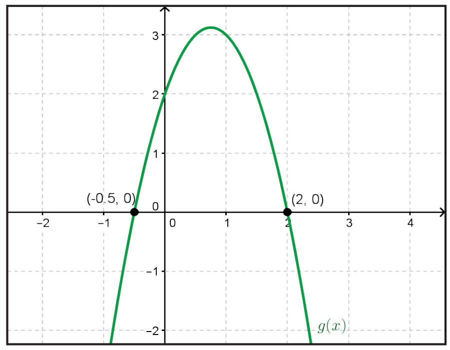
EXAMPLE 2

Given the graph, write a quadratic function $g(x)$ as a product of linear factors. Graph the function you have written and compare it to the given graph. What factor may be missing from your written function?



STEP 1 Use the zeroes in the graph to write the linear factors of the function.

$(-0.5, 0) \rightarrow$ zero: $-\frac{1}{2}$

When a zero is of the form $\frac{a}{b}$, the linear factor is $(bx - a)$, so one linear factor is $(2x - (-1))$ or $(2x + 1)$

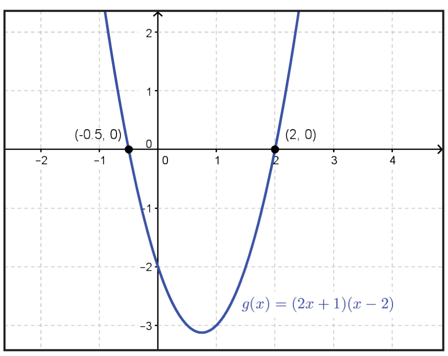
$(2, 0) \rightarrow$ zero: 2

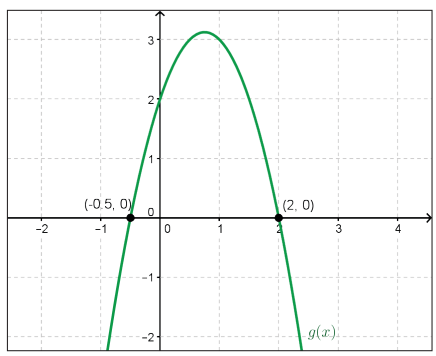
So the other linear factor is $(x - 2)$.

STEP 2 Write the function as a product of its linear factors and as a polynomial.

$g(x) = (2x + 1)(x - 2) = 2x^2 - 4x + 1x - 2 = 2x^2 - 3x - 2$

STEP 3 Graph the written function and compare it to the given graph.





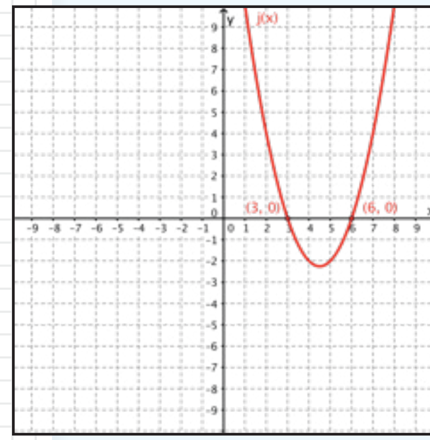
This graph has the same x-intercepts or zeroes, but it is a reflection of the given graph (in green) over the x-axis. This means that a factor of -1 is missing in the written function. Apply this factor to write the function that matches the given graph.

$g(x) = -1(2x + 1)(x - 2)$ or $-1(2x^2 - 3x - 2) = -2x^2 + 3x + 2$

ADDITIONAL EXAMPLES

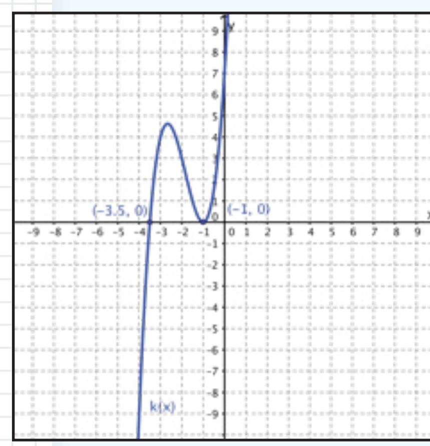
Given the graphs below, write the quadratic or cubic functions as a product of linear factors. Also, using the properties of algebra, write it as a polynomial function.

1.



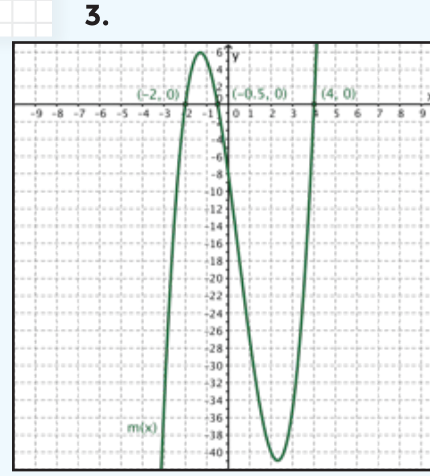
$j(x) = (x - 3)(x - 6) = x^2 - 9x + 18$

2.



$k(x) = (2x + 7)(x + 1)^2 = 2x^3 + 11x2 + 16x + 7$

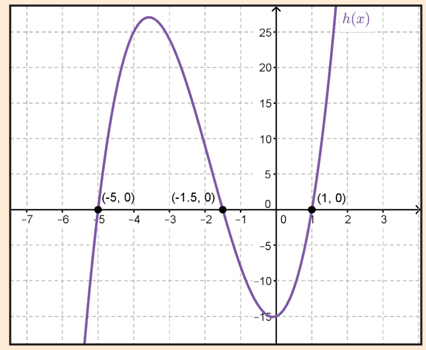
3.



$m(x) = (x + 2)(2x + 1)(x - 4) = 2x^3 - 3x^2 - 18x - 8$

YOU TRY IT! #2

Given the graph of $h(x)$, identify its zeroes and write a cubic function as a product of linear factors.



The zeroes for the function shown in the graph are -5, -1.5 or $-\frac{3}{2}$, and 1. Therefore, $h(x) = (x + 5)(2x + 3)(x - 1)$.

EXAMPLE 3

Write the polynomial function $q(x) = x^3 + 6x^2 + 9x$ as a product of linear factors.

STEP 1 Look for common factors in every term. If there are any, factor those out first. There is a factor of x in each of the terms.

$q(x) = x^3 + 6x^2 + 9x = x(x^2 + 6x + 9)$

STEP 2 Determine if the remaining factor, $(x^2 + 6x + 9)$, follows a pattern of differences of squares or perfect square trinomials.

$x^2 + 6x + 9$ follows a pattern of perfect squares trinomials:
$a^2 + 2ab + b^2 = (a + b)^2$ where a is x and b is 3. So $x^2 + 6x + 9 = (x + 3)^2$.

STEP 3 Write $q(x) = x^3 + 6x^2 + 9x$ as a product of linear factors.

$q(x) = x(x + 3)(x + 3) = x(x + 3)^2$.

ADDITIONAL EXAMPLES

Given the functions below, write the quadratic or cubic functions as a product of linear factors.

1. $n(x) = 27x^3 - 125$

Since $n(x)$ is a difference of cubes, it can be rewritten as $n(x) = (3x - 5)(9x^2 + 15x + 25)$.

2. $p(x) = 16x^2 - 72x + 81$

Since $p(x)$ is a perfect square trinomial, it can be rewritten as $p(x) = (4x - 9)^2$.

3. $q(x) = 64x^3 + 336x^2 + 588x + 343$

Since $q(x)$ is a perfect cube polynomial, it can be rewritten as $q(x) = (4x + 7)^3$.

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YOU TRY IT! #3

Write the polynomial function $p(x) = 25x^2 - 81$ as a product of linear factors.

This function follows the pattern of the difference of squares: p(x) = (5x – 9)(5x + 9).

EXAMPLE 4

Write the cubic function $f(x) = 4x^3 + 12x^2 - x - 3$ as a product of linear factors by grouping its terms and factoring.

STEP 1 **Group the terms of the cubic function and analyze them for common factors.**

$f(x) = (4x^3 + 12x^2) - (x + 3)$. Notice that by grouping the last two terms, you factor -1 from both $-x$ and -3.

STEP 2 **Identify and factor out a common factor in each of the grouped terms.**

$f(x) = 4x^2(x + 3) - 1(x + 3)$

STEP 3 **Factor out the common binomial between the two pairs of factored terms.**

$f(x) = (x + 3)(4x^2 - 1)$

STEP 4 **Determine if either factor can be written as a product of factors.**

$(4x^2 - 1)$ is a difference of two squares. So $(4x^2 - 1)$ can be written as $(2x + 1)(2x - 1)$.

STEP 5 **Write the final factored form of the function: $f(x) = (x + 3)(2x + 1)(2x - 1)$.**

YOU TRY IT! #4

Write the quadratic function $g(x) = 3x^2 + 13x - 10$ as a product of linear factors by finding those pairs of numbers that produce the product of its first and last terms' coefficients and a sum of the middle term's coefficient.

See margin.

ADDITIONAL EXAMPLES

Given the functions below, write the quadratic or cubic functions as a product of linear factors by grouping their terms and factoring.

1. $r(x) = 5x^2 + 13x - 6$

$r(x) = (5x - 2)(x + 3)$

2. $s(x) = 3x^3 + 2x^2 - 75x - 50$

$s(x) = (3x + 2)(x^2 - 25) = (3x + 2)(x - 5)(x + 5)$

3. $t(x) = 4x^3 - 24x^2 + 9x - 54$

$t(x) = (4x^2 + 9)(x - 6)$

YOU TRY IT! #4 ANSWER:

PAIRS WITH PRODUCT OF -30	-1, 30	1, -30	-2, 15	2, -15	-3, 10	3, -10
SUMS	29	-29	13	-13	7	-7

The numbers that have a product of –30 and a sum of 13 are 15 and –2. Write $g(x) = 3x^2 + 13x - 10$ as $g(x) = 3x^2 + 15x - 2x - 10$ to factor as $3x(x + 5) - 2(x + 5)$ and finally $g(x) = (3x - 2)(x + 5)$.

PRACTICE/HOMEWORK

For questions 1 – 4, use the table of values to write a function, f(x), as a product of linear factors. Then, write the function as a polynomial.

1.

x	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
$f(x)$	0	−9	−16	−21	−24	−25	−24	−21	−16	−9	0	11

$f(x) = (x - 5)(x + 5) = x^2 - 25$

2.

x	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
$f(x)$	−140	−72	−30	−8	0	0	−2	0	12	40	90	168

$f(x) = x(x - 2)(x + 1) = x^3 - x^2 - 2x$

3.

x	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
$f(x)$	−54	−16	0	0	−10	−24	−36	−40	−30	0	56	144

$f(x) = (x - 4)(x + 2)(x + 3) = x^3 + x^2 - 14x - 24$

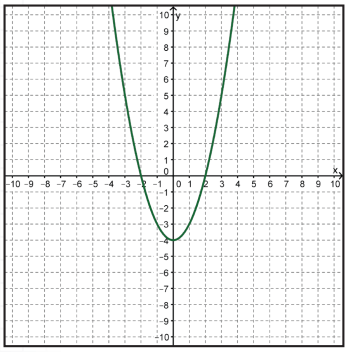
4.

x	−1.25	−1	−0.75	−0.5	−0.25	0	0.25	0.5	0.75	1	1.25
$f(x)$	−21	−9	−2.5	0	0	−1	−1.5	0	5	15	31.5

$f(x) = (2x - 1)(4x + 1)(2x + 1) = 16x^3 + 4x^2 - 4x - 1$

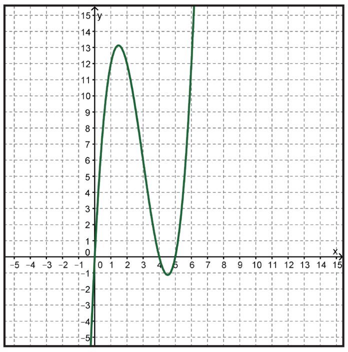
For questions 5 – 8, use the graph to write a function, g(x), as a product of linear factors. Then, write the function as a polynomial.

5.



$g(x) = (x - 2)(x + 2) = x^2 - 4$

6.



$g(x) = x(x - 4)(x - 5) = x^3 - 9x^2 + 20x$

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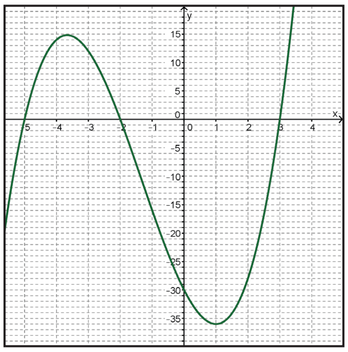
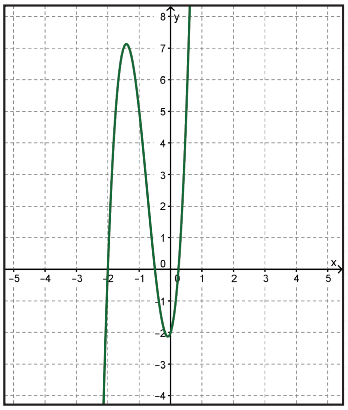
7.



$g(x) = (x - 3)(x + 2)(x + 5) =$
$x^3 + 4x^2 - 11x - 30$

8.



$g(x) = (4x - 1)(2x + 1)(x + 2) =$
$8x^3 + 18x^2 + 3x - 2$

For questions 9 – 14, write the polynomial function as a product of linear factors.

9. $g(x) = x^2 - 9$
$g(x) = (x - 3)(x + 3)$

10. $h(x) = 4x^2 + 16x + 16$
$h(x) = 4(x + 2)^2$

11. $p(x) = x^3 + 10x^2 + 25x$
$p(x) = x(x + 5)^2$

12. $q(x) = 8x^3 + 27$
$q(x) = (2x + 3)(4x^2 - 6x + 9)$

13. $r(x) = 27x^3 - 1$
$r(x) = (3x - 1)(9x^2 + 3x + 1)$

14. $s(x) = x^3 + 4x^2 - 4x - 16$
$s(x) = (x - 2)(x + 2)(x + 4)$

For questions 15 - 20, write the function as a product of linear factors by grouping its terms and factoring.

15. $f(x) = 2x^2 - 7x + 3$
$f(x) = (x - 3)(2x - 1)$

16. $g(x) = 4x^2 + 16x + 15$
$g(x) = (2x + 3)(2x + 5)$

17. $h(x) = 6x^2 - 11x - 10$
$h(x) = (2x - 5)(3x + 2)$

18. $m(x) = x^3 + 6x^2 - 4x - 24$
$m(x) = (x - 2)(x + 2)(x + 6)$

19. $n(x) = x^3 + 8x^2 - 9x - 72$
$n(x) = (x + 3)(x - 3)(x + 8)$

20. $p(x) = x^3 + 4x^2 - 16x - 64$
$p(x) = (x - 4)(x + 4)^2$

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