Applying Polynomial Functions

5.5

FOCUSING QUESTION Which operations on polynomial functions do I need to use in order to solve a real-world problem?

LEARNING OUTCOMES

- I can use polynomial operations to solve a real-world problem.
- I can use a problem-solving model to solve problems involving operations on polynomial functions.

ENGAGE

In 2014, South Carolina produced 65,700 tons of peaches and Georgia produced 35,500 tons of peaches. If the wholesale price of peaches was $975 per ton, how much more money did the South Carolina peach crop generate than the Georgia peach crop?

$29,445,000

Flameprince Peaches, U.S. Dept. of Agriculture, Wikimedia Commons

EXPLORE

The population of a state is important for several reasons. State and local leaders need to know how to allocate resources such as money for road repairs, how many new schools or hospitals should be built, or how much drinking water will be necessary in the future.

The table below shows the population of North Carolina and South Carolina every 5 years, beginning with 1970.

YEAR	NORTH CAROLINA (MILLIONS OF PEOPLE)	SOUTH CAROLINA (MILLIONS OF PEOPLE)
1970	5.08	2.6
1975	5.55	2.9
1980	5.88	3.12
1985	6.25	3.3
1990	6.66	3.5
1995	7.34	3.75
2000	8.08	4.02
2005	8.71	4.27
2010	9.56	4.64
2015	10.04	4.89

Data Source: U.S. Census Bureau

TEKS

AR.4A Connect tabular representations to symbolic representations when adding, subtracting, and multiplying polynomial functions arising from mathematical and real-world situations, such as applications involving surface area and volume.

MATHEMATICAL PROCESS SPOTLIGHT

AR.1B Use a problem-solving model that incorporates analyzing given information, formulating a plan or strategy, determining a solution, justifying the solution, and evaluating the problem-solving process and the reasonableness of the solution.

ELPS

4F Use visual and contextual support and support from peers and teachers to read grade-appropriate content area text, enhance and confirm understanding, and develop vocabulary, grasp of language structures, and background knowledge needed to comprehend increasingly challenging language.

VOCABULARY

linear, quadratic, cubic, sum, addend, rate

MATERIALS

- graphing technology

Let x represent the number of five-year periods since 1970. In 2010, the per capita consumption of water, or total amount of water used divided by the population, in North Carolina and South Carolina was 1,260 gallons each day (source: U.S. Geological Survey). At this rate of consumption, how much water will North Carolina and South Carolina need to meet the needs of their combined population in 2050? Use a problem-solving model to determine your solution.

> In 1945, George Polya wrote *How to Solve It*, a short book describing a four-step problem-solving model. Polya's problem-solving model was:
> I. Understand the problem.
> II. Make a plan.
> III. Carry out the plan.
> IV. Look back on your work.

UNDERSTAND THE PROBLEM: ANALYZE THE GIVEN INFORMATION

1. What information are you given in the problem?
   See margin.

2. What is the problem asking you to do?
   See margin.

Image credit: Wikimedia Commons

MAKE A PLAN: FORMULATE A PLAN OR STRATEGY

3. What is one method you can use to solve this problem?
   See margin.

4. You are given data in a table. What other representation(s) could help you create a plan to solve the problem? How does that representation or those representations help you create a plan?
   See margin.

CARRY OUT THE PLAN: DETERMINE THE SOLUTION

One plan for solving this problem is to use finite differences to write function rules for $N(x)$ and $S(x)$, combine the function rules, then use the combined function rule to predict the population in 2050.

5. Use finite differences to write a function rule for $N(x)$.
   See margin.

6. Use finite differences to write a function rule for $S(x)$.
   See margin.

7. Determine $C(x) = N(x) + S(x)$ to represent the combined population function.
   $C(x) = (0.551x + 5.08) + (0.254x + 2.6) = 0.805x + 7.68$

1. *Answers may vary. Possible answer: We have a table of data showing the population of North Carolina and South Carolina in five-year intervals, beginning in 1970. We also have the daily water consumption for both North Carolina and South Carolina in 2010.*

2. *Answers may vary. Possible answer: Determine the combined population of North Carolina and South Carolina in 2050 and use that population to predict how much water both North Carolina and South Carolina will need.*

3. *Answers may vary. Possible answers: Use finite differences to determine a function rule, N(x), for North Carolina's population and a function rule, S(x), for South Carolina's population. Use the function rules to predict the population in 2050. Then, multiply the population in 2050 by the per capita daily water consumption.*

4. *Answers may vary. Possible answer: Use the data to make a scatterplot. Look to see if the scatterplot has the characteristics of a linear, quadratic, cubic, exponential, or other type of function.*

5. *A scatterplot shows that the data are approximately linear, so calculate the average first difference to use for the function rule.*

   $N(x) = 0.551x + 5.08$

FIVE-YEAR PERIOD SINCE 1970, x	YEAR	NORTH CAROLINA (MILLIONS OF PEOPLE)
0	1970	5.08
1	1975	5.55
2	1980	5.88
3	1985	6.25
4	1990	6.66
5	1995	7.34
6	2000	8.08
7	2005	8.71
8	2010	9.56
9	2015	10.04

+1	+0.47
+1	+0.33
+1	+0.37
+1	+0.41
+1	+0.68
+1	+0.74
+1	+0.63
+1	+0.85
+1	+0.48

6. *A scatterplot shows that the data are approximately linear, so calculate the average first difference to use for the function rule.*

   $S(x) = 0.254x + 2.6$

FIVE-YEAR PERIOD SINCE 1970, x	YEAR	SOUTH CAROLINA (MILLIONS OF PEOPLE)
0	1970	2.6
1	1975	2.9
2	1980	3.12
3	1985	3.3
4	1990	3.5
5	1995	3.75
6	2000	4.02
7	2005	4.27
8	2010	4.64
9	2015	4.89

+1	+0.3
+1	+0.22
+1	+0.18
+1	+0.2
+1	+0.25
+1	+0.27
+1	+0.25
+1	+0.37
+1	+0.25

8. Use $C(x)$ to determine the combined population of both states in 2050 (i.e., $x = 16$ since 2050 is 16 five-year periods after 1970).
C(16) = 0.805(16) + 7.68 = 20.56 million

9. Use the combined population in 2050 and the per capita daily water consumption rate to estimate the total daily water consumption in 2050.
20,560,000 people × 1,260 gallons/day/person = 25,905,600,000 gallons/day

LOOK BACK ON YOUR WORK: JUSTIFY THE SOLUTION AND EVALUATE ITS REASONABLENESS

10. Make a scatterplot of both data sets and graph the function rules over the scatterplot. Describe how the graphs of the function rules match up with the scatterplots for their respective data sets.
See margin.

11. Use graphing technology to combine $N(x)$ and $S(x)$ into $C(x)$. Use your graph to determine the value of $C(x)$, the combined population of both North Carolina and South Carolina, for the year 2050.
See margin.

12. Justify your solution using the equations, tables, or graphs. Include explanations of why you chose the operations on the functions that you did.
See margin.

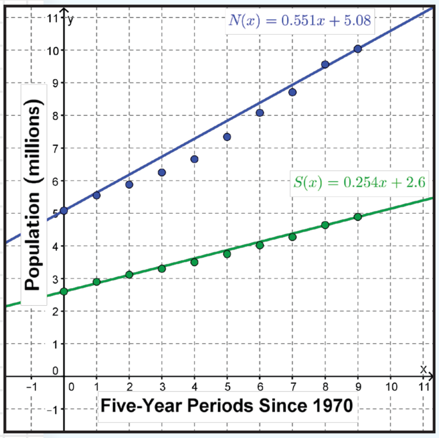
13. Use rounding to estimate the number of gallons of water per day required to support the combined population of both North Carolina and South Carolina in 2050. What does your estimate tell you about the reasonableness of the solution you calculated?
See margin.

REFLECT

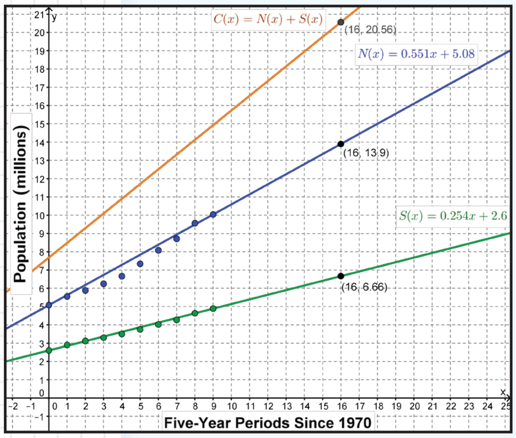
- **How did using a problem-solving process help you to solve the problem?**
  See margin.
- **How can using different representations (e.g., graphs, tables, or equations) help ensure that your solution is reasonable?**
  See margin.

10. *N(x) connects the first and last data point but seems to overestimate the data values near the middle of the data set.*

S(x) connects most of the points.



11. *In 2050 (x = 16), the combined population is about 20.56 million, or 20,560,000, people.*



INTEGRATING TECHNOLOGY

Use the trace feature of graphing technology to identify the function value $C(16)$.

12. *Answers may vary. Possible response: Combining the populations of the two states is addition. If you determine a function rule for the population of each state, you can use the operation of addition and the properties of algebra to combine the two function rules into C(x), which will tell you the total population of both states.*

Per capita water consumption is a rate. Multiply the rate by the number of people in both states in 2050 so that you can determine the amount of water that will be consumed, or used, by all persons in North Carolina and South Carolina each day.

$$\text{Number of People} \times \frac{\frac{\text{Number of Gallons}}{\text{Day}}}{\text{Person}} = \frac{\text{Number of Gallons}}{\text{Day}}$$

13. *Round 20,560,000 to 21,000,000. Round 1,260 gallons per person per day to 1,000 gallons per person per day.*

Multiply 21,000,000 people by 1,000 gallons per person per day.

21,000,000,000 is close to the calculated solution of 25,905,600,000 gallons per day, so the answer is reasonable.

REFLECT ANSWERS:

See margin on page 546

REFLECT ANSWERS:

A problem-solving process provides a structure to help move through each step of the solution. Analyzing the situation to determine what you know and what you need to know helps to formulate a possible plan or strategy. When you are carrying out the plan, you can continually check your solution steps for reasonableness. Once you have a final solution to the problem, then you can look back and use other representations or strategies such as estimation to make sure that your answer is both reasonable and makes sense.

Different representations allow you to determine a solution using different methods. If you can obtain the same solution different ways, then it increases your confidence that your solution is correct.

INSTRUCTIONAL HINT

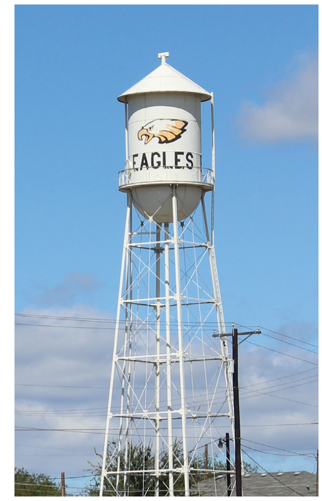
After reading the Explain problem, have students write their four steps to solve this problem. Then allow students to turn to a partner to talk about their plan. Finally have students revise their four steps, if needed.

EXPLAIN

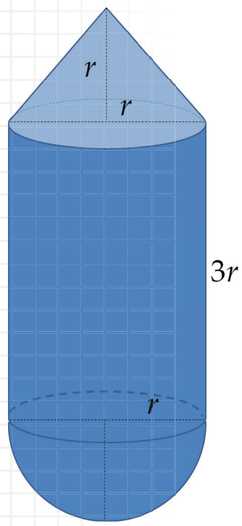
Watch Explain and You Try It Videos

or click here

Some data sets follow patterns for one function. But frequently, you have to add, subtract, or multiply functions together in order to model a situation.



For example, a popular water tower design consists of a cylinder with a cone on top and a hemisphere on bottom. There is no volume function for this composite shape, but you can create one by using the volume functions for each component shape and combining them. For a particular series of water towers, the height and radius of the cone, radius of the cylinder, and radius of the hemisphere are all congruent. If this length is r units, then you can calculate the volume of each component.



- Cylinder: $V = \pi r^2 h$, so $C(r) = \pi r^2(3r) = 3\pi r^3$
- Cone: $V = \frac{1}{3}\pi r^2 h$, so $N(r) = \frac{1}{3}\pi r^2(r) = \frac{1}{3}\pi r^3$
- Hemisphere: $V = \frac{1}{2}(\frac{4}{3}\pi r^3)$, so $H(r) = \frac{2}{3}\pi r^3$

Combining these three functions into one function for the volume of the water tower enables you to write one function that you can use to make predictions and solve problems.

Volume of Water Tower = Volume of Cylinder + Volume of Cone + Volume of Hemisphere

- $V(r) = C(r) + N(r) + H(r)$
- $V(r) = 3\pi r^3 + \frac{1}{3}\pi r^3 + \frac{2}{3}\pi r^3$
- $V(r) = 4\pi r^3$

You can confirm this with a table of values.

RADIUS r	VOLUME OF CYLINDER $C(r) = 3\pi r^3$	VOLUME OF CONE $N(r) = \frac{1}{3}\pi r^3$	VOLUME OF HEMISPHERE $H(r) = \frac{2}{3}\pi r^3$	VOLUME OF WATER TOWER $V(r) = C(r) + N(r) + H(r)$
0	$3\pi(0)^3 = 0$	$\frac{1}{3}\pi(0)^3 = 0$	$\frac{2}{3}\pi(0)^3 = 0$	$0 + 0 + 0 = 0$
1	$3\pi(1)^3 = 3\pi$	$\frac{1}{3}\pi(1)^3 = \frac{1}{3}\pi$	$\frac{2}{3}\pi(1)^3 = \frac{2}{3}\pi$	$3\pi + \frac{1}{3}\pi + \frac{2}{3}\pi = 4\pi$
2	$3\pi(2)^3 = 24\pi$	$\frac{1}{3}\pi(2)^3 = \frac{8}{3}\pi$	$\frac{2}{3}\pi(2)^3 = \frac{16}{3}\pi$	$24\pi + \frac{8}{3}\pi + \frac{16}{3}\pi = 32\pi$
3	$3\pi(3)^3 = 81\pi$	$\frac{1}{3}\pi(3)^3 = 9\pi$	$\frac{2}{3}\pi(3)^3 = 18\pi$	$81\pi + 9\pi + 18\pi = 108\pi$
4	$3\pi(4)^3 = 192\pi$	$\frac{1}{3}\pi(4)^3 = \frac{64}{3}\pi$	$\frac{2}{3}\pi(4)^3 = \frac{128}{3}\pi$	$192\pi + \frac{64}{3}\pi + \frac{128}{3}\pi = 256\pi$
5	$3\pi(5)^3 = 375\pi$	$\frac{1}{3}\pi(5)^3 = \frac{125}{3}\pi$	$\frac{2}{3}\pi(5)^3 = \frac{250}{3}\pi$	$375\pi + \frac{125}{3}\pi + \frac{250}{3}\pi = 500\pi$

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Using finite differences, you can write a function for $V(r)$.

	RADIUS r	VOLUME OF WATER TOWER $V(r)$	First differences	Second differences	Third differences
	0	0			
+1	1	4π	4π		
+1	2	32π	28π	24π	
+1	3	108π	76π	48π	24π
+1	4	256π	148π	72π	24π
+1	5	500π	244π	96π	24π

The third differences are constant, so $V(r)$ is a cubic function.

$$6a = 24\pi \qquad a = 4\pi$$

$$\begin{aligned} 6a + 2b &= 24\pi \\ 6(4\pi) + 2b &= 24\pi \\ 24\pi + 2b &= 24\pi \\ 2b &= 0 \\ b &= 0 \end{aligned}$$

$$\begin{aligned} a + b + c &= 4\pi \\ (4\pi) + (0) + c &= 4\pi \\ 4\pi + c &= 4\pi \\ c &= 0 \end{aligned}$$

$$d = 0$$

Using the values for a, b, c, and d with the general form of a cubic function generates the function $V(r) = 4\pi r^3$.

APPLYING OPERATIONS WITH POLYNOMIAL FUNCTIONS

Addition, subtraction, and multiplication of polynomial functions, including linear, quadratic, and cubic functions, can be used to solve mathematical and real-world problems. When adding, subtracting, or multiplying polynomial functions, you can use one of two methods.

1. Make a table of values for the components to generate the values of the function that is the sum, difference, or product. Then, use finite differences to generate a symbolic function rule.
2. Add, subtract, or multiply the polynomial functions together by adding, subtracting, or multiplying the symbolic function rules and using the properties of algebra to simplify the expressions.

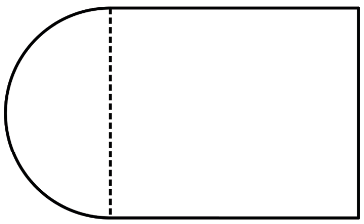
APPLYING OPERATIONS WITH POLYNOMIAL FUNCTIONS

For real-world or mathematical problems for which the solution is not obvious, a problem-solving model can help structure your thinking.

- **Understand the problem.**
  What information does the problem provide?
  What does the problem ask you to determine?
- **Make a plan.**
  What formulas or strategies could you use?
  How could technology help?
- **Carry out the plan.**
  Follow the steps to implement the plan you selected. After each step, pause and make sure your answer is reasonable.
- **Look back on your work.**
  Use different representations to solve the problem another way. Compare the solution to your original solution.Use estimation to see if your calculated answer is reasonable.

EXAMPLE 1

What is the total perimeter of a figure that is composed of a semicircle and a rectangle if its straight edges are created from a single piece of metal 350 inches long?



STEP 1 Understand the problem.

- The figure has a semicircle on its left side and a rectangle on its right side.
- The width of the rectangle on the right side of the figure is the same length as the diameter of the semicircle.
- The straight edges (two lengths and one width of the rectangle) are made from a piece of metal 350 inches long.
- You need to use this information to determine the total perimeter of the figure.

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STEP 2 **Make a plan.**

- If you let x represent the length of the rectangle, then you can use what you know about the length of the single metal piece to represent the width in terms of x.
- Once you have represented the width of the rectangle, you have also represented the diameter of the semicircle because they have the same length.
- You can then use what you know about diameters of circles and their circumference to represent the half of the circle's circumference that makes up the semicircular piece.
- Finally, you can add all the lengths together to determine the total perimeter of the figure.

STEP 3 **Carry out the plan.**

Let x = the length of the rectangle. The polynomial $w(x)$ will be its width.

If the length of the straight edges altogether is 350, then

$$\begin{aligned} 350 &= x + w(x) + x \\ 350 &= w(x) + x + x \\ 350 &= w(x) + 2x \\ 350 - 2x &= w(x) + 2x - 2x \\ 350 - 2x &= w(x) \end{aligned}$$

Therefore, the diameter of the semicircle is $350 - 2x$. The polynomial $c(x)$ will be half the circumference of a circle with the same diameter as the semicircle.

Circumference = pi · diameter

$$c(x) = \frac{1}{2}(\pi)(350 - 2x) = \frac{350\pi - 2\pi x}{2} = 175\pi - \pi x$$

Now, you can add the semicircular piece to the width and lengths to determine a representation for $p(x)$, the perimeter of the figure.

$$p(x) = x + x + w(x) + c(x) = 2x + (350 - 2x) + (175\pi - \pi x)$$
$$p(x) = 350 + 175\pi - \pi x$$

ADDITIONAL EXAMPLE

When Savannah got a new job as a teacher, she opened a new checking account and planed her budget. She initially deposited \$1000 into the checking account, and each month after that she deposits her \$3045 paycheck. Each month Savannah must withdraw \$2560 for her bills. The money left in her checking account is for emergencies and entertainment. What is the total amount of money left in Savannah's account at any given month after she started her new job?

Understand the problem.
Savannah opened a checking account by depositing \$1000. Every month after that starting deposit, she adds \$3045 from her paycheck. Each month she also pays bills for a total of \$2560. Any leftover money is hers to use as she would like. You are asked to write a polynomial to help figure out how much extra money she has in her checking account at the end of any month.

Make a plan.
Use the scenario or a table of values to generate polynomials for $d(x)$, the amount of money that Savannah deposits into her checking account over x months, and $w(x)$, the amount of money that she withdraws from her account each month. Then subtract the two polynomials together to find $b(x)$, the balance left in Savannah's checking account at the end of each month after she deposits her paycheck and withdraws money for her bills.

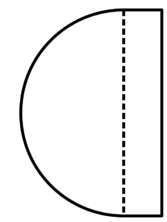
Carry out the plan.
Using a table of values or the scenario, write the polynomials. If students used a table of values, they would use finite differences to write the two linear polynomials.
$d(x) = 1000 + 3045x$
$w(x) = 2650x$
$b(x) = d(x) - w(x) = (1000 + 3045x) - (2650x) = 1000 + (3045 - 2650)x = 1000 + 395x$

Look back at your work.
Using reasonable values, we can test the expression for $b(x)$ using $x = 10$.
$b(x) = 1000 + 395(10) = \$4950$
If $b(x) = 1000 + 395x$, then Savannah saves an additional \$395 per month, about \$400. That means she saves about 10(400) = \$4000 in 10 months. Since \$4950 is almost \$5000, this test shows our polynomial is reasonable.

STEP 4 **Look back on your work. Justify your solution and determine if your solution is reasonable.**

The figure that is cut from a sheet of metal is a composite figure made up of a rectangle and a semicircle. Perimeter is an additive property of geometric figures, so the combined perimeter of the figure is the circumference of the semicircle added to the edges of the rectangle that are on the finished edges of the figure.

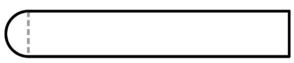
The perimeter of the figure will depend on how long the length of the rectangle is. You can look at extreme examples to determine if your answer is reasonable.



If the length of the rectangular part of the figure is 10 inches, then the width of the rectangle is very long, and so the semicircle will be very large, making the perimeter of the figure quite large. You can evaluate $p(10)$ to determine if the output value is quite large.

$p(x) = 350 + 175\pi - \pi x$
$p(10) = 350 + 175\pi - \pi(10) = 350 + 165\pi \approx 868.363$

The perimeter is approximately 868 inches, which is quite large.



If the length of the rectangular part of the figure is 170 inches, then the width of the rectangle is very short, and so the semicircle will be very small, making the perimeter of the figure close to 350. You can evaluate $p(170)$ to determine if the output value is close to 350, the length of the piece that made the straight edges.

$p(x) = 350 + 175\pi - \pi x$
$p(170) = 350 + 175\pi - \pi(170) = 350 + 5\pi \approx 365.708$

The perimeter is close to 350.

Therefore, $p(x)$ is a reasonable representation of the perimeter of the figure.

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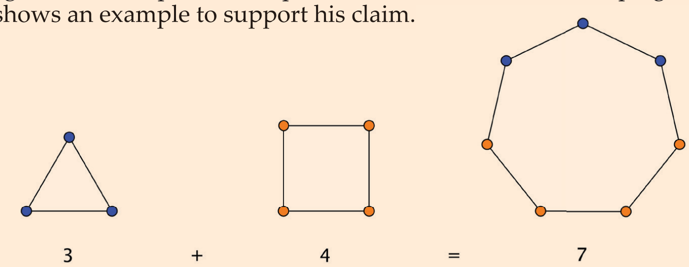
YOU TRY IT! #1

Antonio studies figurate numbers and thinks he has found a pattern. He claims that each triangular number plus each square number results in each heptagonal number. Antonio shows an example to support his claim.

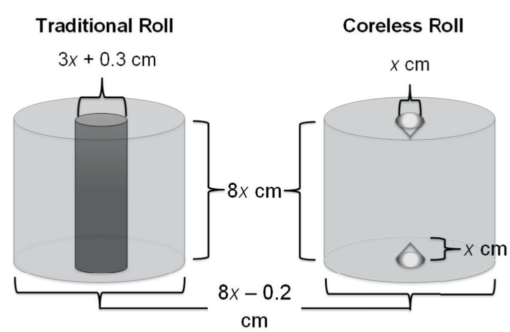


Write polynomials for each type of figurate number to determine whether the pattern Antonio noticed is true for every triangular, square, and heptagonal figurate number. Use the problem solving process, justify your solution, and determine if your solution is reasonable.

See margin.

EXAMPLE 2

A paper company offers two different two-ply toilet paper rolls. The first is a traditional roll with a cardboard tube. The second is a coreless roll that is completely made of toilet paper and only has two small cone-shaped indentations on each side of the roll for dispenser placement. The paper company advertises the coreless roll as having up to twice the capacity of a traditional roll of toilet paper. Evaluate the company's claim.



STEP 1 Understand the problem.

The paper company offers traditional cylindrical toilet paper rolls, pictured on the left side of the diagram, and coreless cylindrical toilet paper rolls, pictured on the right side of the diagram. The traditional roll has a cardboard tube. There is not toilet paper in that part of the cylinder. The coreless roll has two little cones on each side where the dispenser holds onto the roll. Both types of toilet paper roll have the same height and outside diameter. You are asked to determine whether or not the company's claim that the coreless roll has up to twice the capacity of the traditional roll is true.

YOU TRY IT! #1 ANSWER:

Understand the problem.

Antonio found one case where a triangular number plus a square number equaled a heptagonal number. The problem asks if this relationship is true for every triangular, square, and heptagonal number in a sequence.

Make a plan.

Make tables to represent the figurate number patterns and use the patterns in the tables to write polynomials to represent the figurate numbers. Let n represent the position in the sequence of figurate numbers. Determine if the sum of the triangular number function and the square number function is equal to the heptagonal number function.

Carry out the plan.

- *Triangular numbers: 1, 3, 6, 10,… can be represented by the function* $T(n) = \frac{1}{2}n^2 + \frac{1}{2}n$.
- *Square numbers: 1, 4, 9, 16,… can be represented by the function* $S(n) = n^2$.
- *Heptagonal numbers 1, 7, 18, 34,… can be represented by the polynomial function* $H(n) = \frac{5}{2}n^2 - \frac{3}{2}n$.

If the pattern Antonio noticed is true for all triangular, square, and heptagonal numbers, then the heptagonal numbers will be represented by $(T + S)(n) = \frac{3}{2}n^2 + \frac{1}{2}n$. *Since* $H(n) \neq (T + S)(n)$, *the pattern Antonio noticed is not true for every triangular, square, and heptagonal figurate number.*

Look back at your work.

Finite differences can be used to determine the functions T(n), S(n), and H(n). Functions can be added together by combining like terms and applying the properties of algebra. The function (T + S)(n) can be compared to H(n) for equivalence. Since (T + S)(n) and H(n) are not equivalent, the pattern Antonio noticed is not true for all figurate numbers.

To evaluate the solution for reasonableness, select a different set of figurate numbers. The third numbers in each sequence are $T(3) = 6$, $S(3) = 9$, *and* $H(3) = 18$. $6 + 9 = 15$, *which is not equal to 18. Therefore, for the third set of figurate numbers,* $T(3) + S(3) \neq H(3)$ *and is a counterexample for Antonio's pattern.*

STEP 2 **Make a plan.**

To find the amount of toilet paper in each roll, you need to determine the volume of each roll. The formulas for volume of a cylinder and volume of a cone will be helpful.

Volume of a Cylinder = (Area of the Base)(Height of the Cylinder)
$V = Bh = \pi r^2 h$

Volume of a Cone = $\frac{1}{3}$(Area of the Base)(Height of the Cone)
$V = \frac{1}{3}Bh = \frac{1}{3}\pi r^2 h$

Since all the measurements for the rolls are given in terms of x, our volumes will be polynomials.

To determine the volume of the traditional roll, you need to determine the volume of the whole roll and subtract the volume taken up by the cardboard tube. To determine the volume of the coreless roll, you need to determine the volume of the whole roll and subtract the volume taken up by the cone-shaped indentations. Finally, to evaluate the company's claim, you will need to compare the volumes of the two toilet paper rolls.

STEP 3 **Carry out the plan.**

Since both rolls have the same outside dimensions, the volume of the whole roll will be the same in both cases.

Let $W(x)$ represent the volume of the whole roll.
$W(x) = Bh$

$W(x) = \pi r^2 h = \pi\left(\frac{8x - 0.2}{2}\right)^2(8x) = 8\pi x(4x - 0.1)^2 = 8\pi x(16x^2 - 0.8x + 0.01)$

$W(x) = 128\pi x^3 - 6.4\pi x^2 + 0.08\pi x$

Let $b(x)$ represent the volume of the cardboard tube.
$b(x) = Bh$

$b(x) = \pi r^2 h = \pi\left(\frac{3x + 0.3}{2}\right)^2(8x) = 8\pi x(1.5x - 0.15)^2 = 8\pi x(2.25x^2 + 0.45x + 0.0225)$

$b(x) = 18\pi x^3 + 3.6\pi x^2 + 0.18\pi x$

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Let $T(x)$ represent the volume of the traditional roll.
$T(x) = W(x) - b(x)$

$$T(x) = (128\pi x^3 - 6.4\pi x^2 + 0.08\pi x) - (18\pi x^3 + 3.6\pi x^2 + 0.18\pi x)$$

$$T(x) = 128\pi x^3 - 6.4\pi x^2 + 0.08\pi x - 18\pi x^3 - 3.6\pi x^2 - 0.18\pi x$$

$$T(x) = 128\pi x^3 - 18\pi x^3 - 6.4\pi x^2 - 3.6\pi x^2 + 0.08\pi x - 0.18\pi x$$

$$T(x) = 110\pi x^3 - 10\pi x^2 - 0.1\pi x$$

Let $n(x)$ represent the volume of the cone-shaped indentations.

$$n(x) = 2\left(\frac{1}{3}Bh\right) = 2\left(\frac{1}{3}\pi r^2 h\right) = \frac{2}{3}\pi\left(\frac{x}{2}\right)^2(x) = \frac{2}{3}\pi\left(\frac{x^2}{4}\right)(x) = \frac{1}{6}\pi x^3$$

Let $C(x)$ represent the volume of the coreless roll.
$C(x) = W(x) - n(x)$

$$C(x) = (128\pi x^3 - 6.4\pi x^2 + 0.08\pi x) - \left(\frac{1}{6}\pi x^3\right)$$

$$C(x) = 128\pi x^3 - \frac{1}{6}\pi x^3 - 6.4\pi x^2 + 0.08\pi x$$

$$C(x) = \frac{767}{6}\pi x^3 - 6.4\pi x^2 + 0.08\pi x$$

STEP 4 **Look back on your work.**

It is advisable to use reasonable values for x to evaluate the company's claim. If $x = 1$ cm,

$$T(1) = 110\pi(1)^3 - 10\pi(1)^2 - 0.1\pi(1) = 110\pi - 10\pi - 0.1\pi = 99.9\pi$$

and

$$C(1) = \frac{767}{6}\pi(1)^3 - 6.4\pi(1)^2 + 0.08\pi(1) = \frac{767}{6}\pi - 6.4\pi + 0.08\pi \approx 121.513\pi$$

The volume of the traditional roll of toilet paper is approximately 100π cubic centimeters. The coreless toilet paper roll falls well short of having twice the amount of toilet paper as the traditional roll. However, the paper company's claim was that the coreless roll has "up to" twice the amount as the traditional roll. Technically, the paper company's claim is not false.

ADDITIONAL EXAMPLE

A snack company sells a square pyramid shaped box of animal crackers. The sides of its base can be represented by x inches, and its height is three times its base edge length. This box shape is very popular, but the company feels it is inefficient for both the packing space required for shipping and amount of animal crackers that one box can contain. They create a box shaped like a rectangular prism with the same base dimensions as the square pyramid and a height that is three times its base edge length. Since the new box shape is not as eye-catching, the company markets the new container as holding "200% more than the original box." Evaluate the company's claim.

Using the formulas for volume of a pyramid and volume of a prism, the volumes for each container of animal crackers are represented by:

Volume of a pyramid, $p(x)$:
$p(x) = \frac{1}{3}Bh = \frac{1}{3} \cdot x \cdot x \cdot 3x = x^3$

Volume of prism, $r(x)$:
$r(x) = Bh = x \cdot x \cdot 3x = 3x^3$

Test the answer with $x = 6$ to evaluate the claim:

$p(6) = 6^3 = 216 \text{ in}^3$

$r(6) = 3 \cdot 6^3 = 648 \text{ in}^3$

The company has over-estimated how much the volume has increased. It is three times larger, which is equal to its 200% claim.

YOU TRY IT! #2 ANSWER:

Using the patterns in the finite differences in the table, the amount of principal Amanda has invested can be represented as $f(x) = 500x$. The amount of loss Amanda's investment has experienced has constant third differences and so it can be represented by the cubic function $g(x) = -3x^3 + 24x^2 + 60x$. The worth of Amanda's investment is $(f - g)(x) = 3x^3 - 24x^2 + 440x$. The long-term prospect for her high-risk investment is good, since it is described by an increasing cubic function. If the table is continued, the investment loss eventually decreases and the investment starts to make a profit after more than 10 months. Amanda should continue to invest if she plans to invest long-term.

YOU TRY IT! #2

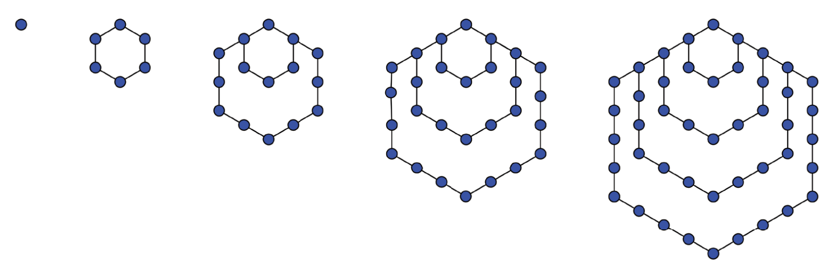
Amanda invests $500 monthly in a high-risk investment because it potentially has a high reward. The investment loses money during the first six months. Amanda tracks the total amount invested and the total amount lost over the first six months of her investment in a table. What is Amanda's investment worth? Should Amanda continue to pay into this high-risk investment? Justify your response.

MONTHS INVESTED, x	PRINCIPAL INVESTED, $f(x)$	INVESTMENT LOSS, $g(x)$
0	$0	$0
1	$500	$81
2	$1,000	$192
3	$1,500	$315
4	$2,000	$432
5	$2,500	$525
6	$3,000	$576

See margin.

EXAMPLE 3

Kelli was exploring hexagonal numbers but found that she had difficulty counting the number of points to determine each hexagonal number in the sequence.



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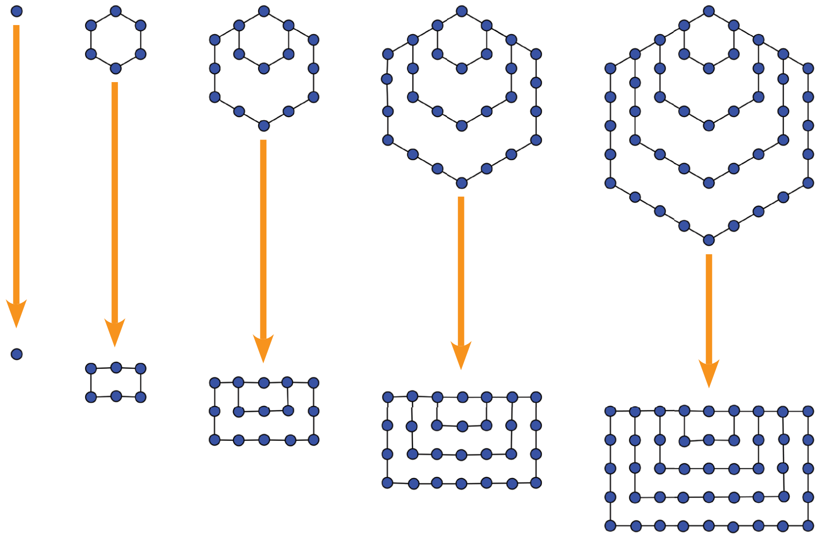
After studying the graphic, Kelli noticed that she could visualize the configuration of points in a different way:



Use a problem-solving model and this new configuration of points to help Kelli determine the 100^{th} hexagonal number.Justify your solution and evaluate the problem-solving process.

STEP 1 **Understand the problem.**

Kelli has determined that she can represent each hexagonal number as a rectangular array of dots. You are asked to use this new representation to determine the 100^{th} hexagonal number.

STEP 2 **Make a plan.**

You can set up a table to represent n, the position in the sequence of hexagonal numbers; $w(n)$, the number of dots in the width of the rectangular array; and $l(n)$, the number of dots in the length of the rectangular array. Then you can use the patterns in the finite differences to determine polynomial representations for $w(n)$ and $l(n)$ and multiply them to determine $(l \cdot w)(n)$, the value of each hexagonal number in the sequence. Finally, you can evaluate $(l \cdot w)(100)$ to determine the value of the 100^{th} hexagonal number.

STEP 3 Carry out the plan.

POSITION IN THE SEQUENCE, n	NUMBER OF ROWS (WIDTH) IN THE RECTANGULAR ARRAY, $w(n)$	NUMBER OF COLUMNS (LENGTH) IN THE RECTANGULAR ARRAY, $l(n)$
1	1	1
2	2	3
3	3	5
4	4	7
5	5	9

It is clear from the table that $w(n) = n$. You can use the patterns in the finite differences to determine that $l(n) = 2n - 1$.

$(l \cdot w)(n) = l(n) \cdot w(n)$
$(l \cdot w)(n) = (2n - 1)(n)$
$(l \cdot w)(n) = 2n^2 - n$

$(l \cdot w)(100) = 2(100)^2 - 100 = 2(10{,}000) - 100 = 20{,}000 - 100 = 19{,}900$
The 100th hexagonal number is 19,900.

STEP 4 Look back on your work.

The rectangular array for the 100th hexagonal number would have a width of 100 and a length of 199. Therefore, the 10th hexagonal number represented as a rectangular array would have 19,900 dots.

JUSTIFICATION OF THE SOLUTION:
Finite differences can be used from a table of data to determine function rules. In this situation, both the number of rows and number of columns in the rectangular array have a constant first differences and are linear functions.

The product of two linear functions is a quadratic function. Hence, $(l \cdot w)(n)$ should be a quadratic function, and our work verifies that it is. Once the function $(l \cdot w)(n)$, it can be evaluated for $n = 100$ for the 100th hexagonal number.

EVALUATION OF THE PROBLEM-SOLVING PROCESS:
In this case, the problem-solving process helped you to organize your thinking so that you could understand the problem, compare different ways to solve it, select a plan, and carry out the plan as you frequently think about whether or not the plan is going in the expected direction, and then look back on your work so that you can make sure your solution is justified, reasonable, and makes sense.

ADDITIONAL EXAMPLES

1. What is the perimeter of a rectangle if its length is 4 more than 2 times its width?
Let $w(x) = x$ represent the width of the rectangle. The length of the rectangle can be represented by $l(x) = 2x + 4$. The perimeter of the rectangle is $2((w + l)(x)) = 2(x + 2x + 4) = 2(3x + 4) = 6x + 8$.

2. What is the area of a triangle if its height is 2 less than 2.5 times its base?
Let $b(x) = x$ represent the base of the triangle. The height of the triangle can be represented by $h(x) = 2.5x - 2$. The area of the triangle is $\frac{1}{2}((b \cdot h)(x)) = \frac{1}{2}(x(2.5x - 2)) = \frac{1}{3}(2.5x^2 - 2x) = 1.25x^2 - x$.

3. What is the volume of a cylinder if its height is 6 times its diameter?
Let $d(x) = x$ represent the diameter of the cylinder. The height of the cylinder can be represented by $h(x) = 6x$. The volume of the cylinder is $\pi(r^2)(h) = \pi\left(\left(\frac{x}{2}\right)^2(6x)\right) = \pi\left(\frac{x^2}{4}\right)(6x) = \frac{3}{2}\pi x^3$.

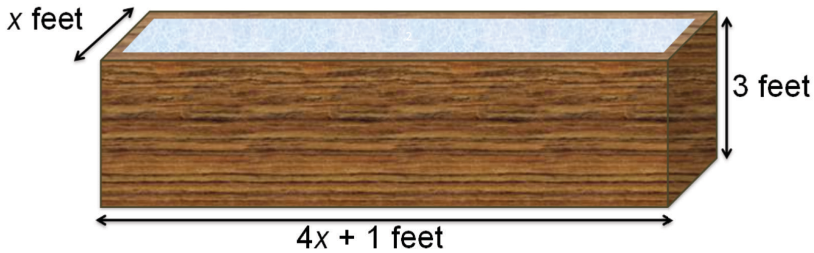
YOU TRY IT! #3

What is the area of the parallelogram if its base is 0.4 meters less than four times as long as its height?

See margin.

EXAMPLE 4

A rancher has an old watering trough that holds approximately 78 cubic feet of water. He wants to replace the trough because it is too small for his growing herd. He asks a welder to make a new watering trough for his animals. The welder's plan is shown below. How long should the new watering trough be to approximately double the volume of the original watering trough?



STEP 1 Understand the problem.

A rancher has a watering trough that holds about 78 cubic feet of water. He wants a new one that holds approximately double that, or 156 cubic feet of water. A welder has plans for one that is three feet high with an indeterminate but related width and length. You are asked to determine how long the new watering trough should be to have a volume of approximately 156 cubic feet.

YOU TRY IT! #3 ANSWER:

Let $h(x) = x$ represent the height of the parallelogram. The length of its base can be represented by $b(x) = 4x - 0.4$. The area of the parallelogram is $(b \cdot h)(x) = 4x^2 - 0.4x$.

STEP 2 **Make a plan.**

You will need to determine the total volume of the watering trough by multiplying its length, width, and height. Then you can set the volume equal to 156 cubic feet and solve for x. Finally, evaluate the polynomial that represents the length of the watering trough for the solved for value of x.

STEP 3 **Carry out the plan.**

Let $V(x)$ represent the volume of the new watering trough.
$V(x) = (4x + 1)(x)(3) = 12x^2 + 3x$

$$156 = 12x^2 + 3x$$
$$12x^2 + 3x - 156 = 0$$
$$(12x^2 + 3x - 156) \div 3 = 0 \div 3$$
$$4x^2 + x - 52 = 0$$

Using the quadratic formula to solve the quadratic equation,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{(1)^2 - 4(4)(-52)}}{2(4)} = \frac{-1 \pm \sqrt{833}}{8}$$

$x = \frac{-1 - \sqrt{833}}{8} \approx -3.733$ or $x = \frac{-1 + \sqrt{833}}{8} \approx 3.483$

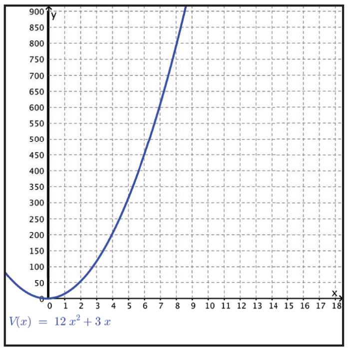
Only the positive value of x makes sense in this situation since x represents the width of the watering trough.

The length of the watering trough is $4x + 1$.
$4(3.483) + 1 \approx 14.932$

The new watering trough should be approximately 15 feet long to have to rancher's desired approximate volume of 156 cubic feet.

STEP 4 **Look back on your work.**

If you use graphing technology to graph $V(x)$, then you see that when x is approximately 3.483, $V(x)$ is approximately 155. Therefore, another representation confirms that your answer is reasonable.



ADDITIONAL EXAMPLE

Eric's sandbox is in the shape of a rectangular prism and holds 50 cubic feet of sand. However, lately Eric's mom has noticed that he needs more space for his sand creations. She asked Eric's dad to build a new sandbox that will hold at least three times as much sand as the old sandbox. In his plan, the new height is 2 feet, and its width is 3 feet more than half of its length. How long should Eric's dad make the new sandbox?

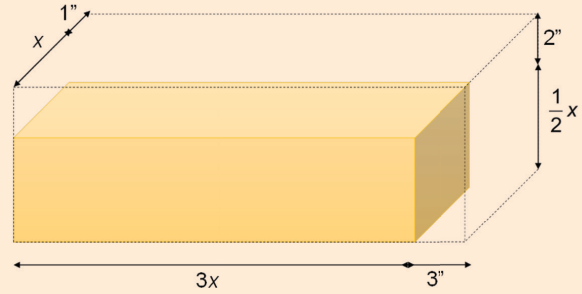
Let $h(x) = 2$ represent the height of Eric's sandbox, $l(x) = x$ represent the length, and $w(x) = \frac{1}{2}x + 3$ represent the width. The volume of Eric's sandbox is $V(x) =$ (area of the base)(height) $= (l \cdot w \cdot h)(x) = x \cdot (\frac{1}{2}x + 3) \cdot 2 = x^2 + 6x$.

To increase the volume as Eric's mother requested, multiply the original volume by 3.
$V(x) = 3 \cdot 50 = 150$
$150 = x^2 + 6x$
$x \approx 9.61$ or $x \approx -15.6$

Only positive values of x make sense in this situation since x represents the length of the sandbox. Therefore,Eric's dad should make the new sandbox about 10 feet long to have at least three times as much sand.

YOU TRY IT! #4

Sean's toy box is too small to fit all of his toys. His mother commissions a carpenter to build a new toy box. She does not take Sean's original toy box to the carpenter but describes it to the carpenter as about three times as long as it is wide and about half as high as it is wide. She asks the carpenter to increase the capacity of the toy box by at least 40%. The carpenter draws up the plans seen below with x representing the depth of Sean's original toy box in inches. Will the carpenter's plan meet Sean's mother's request if the width of Sean's original toy box is 16 inches? Justify your solution and evaluate the problem-solving process.



See margin.

PRACTICE/HOMEWORK

For questions 1 – 15 use the information given in the table below.

CRITICAL THINKING

The tables below show the per capita consumption of bottled water and soft drinks in the U.S. from 2003 to 2013.

U.S. BOTTLED WATER PER CAPITA CONSUMPTION

YEAR	GALLONS PER CAPITA
2003	21.6
2004	23.2
2005	25.4
2006	27.8
2007	29.0
2008	28.5
2009	27.6
2010	28.3
2011	29.2
2012	30.8
2013	32.0

Source: Beverage Marketing Corporation

U.S. SOFT DRINKS PER CAPITA CONSUMPTION

YEAR	GALLONS PER CAPITA
2003	52.4
2004	52.4
2005	51.9
2006	51.0
2007	49.4
2008	47.5
2009	46.1
2010	45.5
2011	44.6
2012	43.6
2013	42.1

Source: The Statistics Portal

YOU TRY IT! #4 ANSWER:

The volume of Sean's original toy box can be represented by $V(x) = (\frac{1}{2}x)(x)(3x) = 1.5x^3$.
To increase its volume by 40%, multiply $V(x)$ *by 140%.*

$1.4V(x) = 2.1x^3$.
If the original width is 16 inches, then the volume of the new toy box must be at least $1.4V(16) = 2.1(16)^3 = 8{,}601.6$ *cubic inches.*

The volume of the carpenter's plan for Sean's new toy box can be represented by

$N(x) = (\frac{1}{2}x + 2)(x + 1)(3x + 3) = 1.5x^3 + 9x^2 + 13.5x + 6$.
$N(16) = 8{,}670$ *cubic inches.*

Yes, the carpenter's plan will meet Sean's mother's request if the width of Sean's original toy box is 16 inches because $8{,}670 > 8{,}601.6$.

1. Let x represent the number of years since 2003, $W(x)$ represent the gallons per capita of bottled water consumed, and $S(x)$ represent the gallons per capita of soft drinks consumed. Complete each of the tables below. Each one has been started for you.

U.S. BOTTLED WATER PER CAPITA CONSUMPTION

x	ACTUAL YEAR	GALLONS PER CAPITA, $W(x)$
0	2003	21.6
1	2004	23.2
2	*2005*	*25.4*
3	*2006*	*27.8*
4	*2007*	*29.0*
5	*2008*	*28.5*
6	*2009*	*27.6*
7	*2010*	*28.3*
8	*2011*	*29.2*
9	*2012*	*30.8*
10	*2013*	*32.0*

U.S. SOFT DRINKS PER CAPITA CONSUMPTION

x	ACTUAL YEAR	GALLONS PER CAPITA, $S(x)$
0	2003	52.4
1	2004	52.4
2	*2005*	*51.9*
3	*2006*	*51.0*
4	*2007*	*49.4*
5	*2008*	*47.5*
6	*2009*	*46.1*
7	*2010*	*45.5*
8	*2011*	*44.6*
9	*2012*	*43.6*
10	*2013*	*42.1*

2. What type of functions do $W(x)$ and $S(x)$ appear to be?
Linear Quadratic Cubic Exponential

3. Describe the difference(s) between $W(x)$ and $S(x)$.
See margin.

4. What do $W(x)$ and $S(x)$ have in common?
See margin.

5. Use finite differences to write a function rule for $W(x)$.
W(x) = 0.99x + 21.6

6. Use finite differences to write a function rule for $S(x)$.
S(x) = -1.03x + 52.4

7. Use graphing technology to make a scatterplot of both data sets and graph the function rules over the scatterplot. How do the function rules match up with their respective data sets?
See margin.

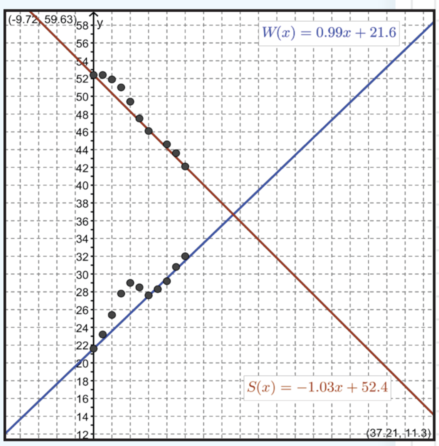
8. To find the combined consumption of bottled water and soft drinks, $C(x)$, write a function rule for $C(x) = W(x) + S(x)$. Simplify the function $C(x)$.
C(x) = -0.04x + 74

9. Use graphing technology to graph $W(x)$, $S(x)$, and $C(x)$ onto the same graph. Based on the graph, what would be the combined consumption of bottled water and soft drinks, $C(x)$ in 2023?
See margin.

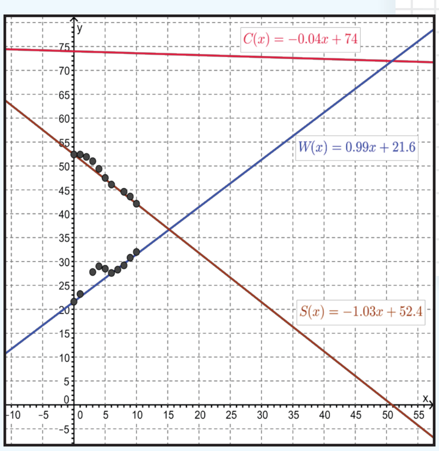
3. *W(x) is an increasing function. S(x) is a decreasing function. The y-intercepts are different; 21.6 for bottled water and 52.4 for soft drinks.*

4. *They are both linear. The rate of change for bottled water is close to the opposite of the rate of change for soft drinks.*

7. *See scatterplot below. The function rules align with most of the data points of each set, however there are approximately 4 points in each data set that are not as aligned due to increase/decrease in consumption.*



9. *Between 70 and 75 gallons per capita*



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10. Based on the function, what would be the combined consumption of bottled water and soft drinks, $C(x)$ in 2023?
Approximately 73.2 gallons per capita

11. In what year will bottled water and soft drink consumption be about the same?
2018

12. If the trend in bottled water consumption continues at the same rate, what is a reasonable estimate of the number of gallons of bottled water consumed per capita in 2023?
Approximately 41.4 gallons

13. If the trend in soft drink consumption continues at the same rate, what is a reasonable estimate of the number of gallons of soft drinks consumed per capita in 2023?
Approximately 31.8 gallons

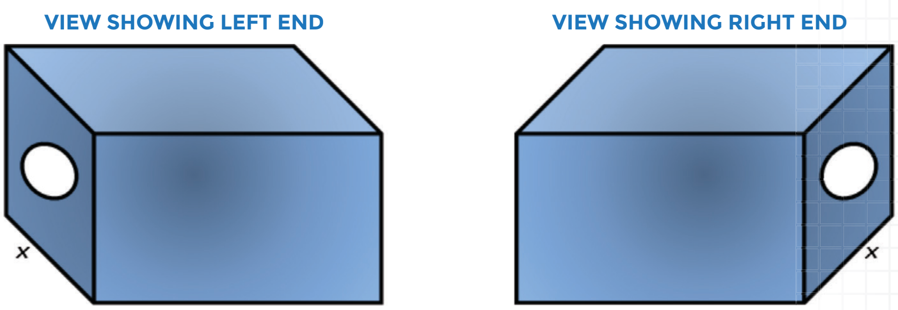
14. Why does $C(x)$ appear to be an approximately horizontal line?
See margin.

15. If the trend in soft drink consumption continues at the same rate, when will people in the U.S. stop drinking soft drinks?
2054

For questions 16 – 25 use the situation below.

GEOMETRY

A manufacturing company is creating a series of boxes from sheet metal. The boxes are rectangular prisms. The length of the base of the prism is twice the width of the base. The height of the prism is two feet less than three times the width of the base. In each end of the prism, there is a circular opening whose diameter is half the width of the base.



16. If x represents the width of the base of the box, represent each of the other dimensions of the box in terms of x.
A. Length of the base of the box ***2x***
B. Height of the box ***3x – 2***
C. Diameter of the circular opening $\frac{1}{2}x$

14. *The combined consumption of bottled water and soft drinks is fairly constant over the years, even though the amount of bottled water is increasing and the amount of soda is decreasing.*

17. Write a function, $S(x)$, to represent the total amount of sheet metal needed to construct the box before removing the circular areas (total surface area) in terms of x.
$S(x) = Ph + 2B$
$S(x) = 22x^2 - 12x$

18. Write a function, $C(x)$, to represent the area of one of the circular openings in terms of x. (Use 3.14 for π)
$C(x) = \pi r^2$
$C(x) = 0.19625x^2$

19. Write a function, $T(x)$, to represent the total areas of the two circular openings.
$T(x) = 0.3925x^2$

20. Use the functions $S(x)$ and $T(x)$ to complete a table like the one shown.

WIDTH IN FEET, x	1	2	3	4	5	6	7	8	9	10
$S(x)$	*10*	*64*	*162*	*304*	*490*	*720*	*994*	*1312*	*1674*	*2080*
$T(x)$	*0.3925*	*1.57*	*3.5325*	*6.28*	*9.8125*	*14.13*	*19.2325*	*25.12*	*31.7925*	*39.25*

21. The amount of sheet metal needed to build a box is the total amount of metal needed, $S(x)$, minus the area of the two circular openings, $T(x)$. Use this information to fill in the last row of the table, $B(x)$.

WIDTH IN FEET, x	1	2	3	4	5	6	7	8	9	10
$S(x)$	*10*	*64*	*162*	*304*	*490*	*720*	*994*	*1312*	*1674*	*2080*
$T(x)$	*0.3925*	*1.57*	*3.5325*	*6.28*	*9.8125*	*14.13*	*19.2325*	*25.12*	*31.7925*	*39.25*
$B(x)$	*9.6075*	*62.43*	*158.4675*	*297.72*	*480.1875*	*705.87*	*974.7675*	*1286.88*	*1642.2075*	*2040.75*

22. Based on finite differences, what type of function is $B(x)$?
Linear Quadratic Cubic Exponential

23. If $S(x)$ represents the total amount of sheet metal needed to construct a box with a width of x, and $T(x)$ represents the total areas of the two circular openings, write a function rule $B(x)$ to represent the amount of sheet metal contained in a box with two circular openings.
$B(x) = 21.6075x^2 - 12x$

24. How much sheet metal is contained in a box with a width of 2 feet?
62.43 square feet

25. How much metal is contained in a box with a width of 12 feet?
2967.48 square feet

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