Comparing Addition and Multiplication of Linear Functions

5.4

FOCUSING QUESTION How is the sum of two linear functions similar to the product of two linear functions? How are the sum and product different?

LEARNING OUTCOMES

- I can add or multiply two linear functions together to make a new function.
- I can compare and contrast the sum or product of two linear functions, including comparing the sum to both addends and the product to both factors.
- I can create and use tables, graphs, and equations to organize, record, and communicate mathematical ideas.

ENGAGE

The High Line park in New York City makes use of railroad ties and rails in landscaping designs. If the park designer has 50 feet of rails to use for outlining a rectangular rock feature, what is the range of areas that the rock feature could have?

$0 < A \leq 156.25$ square feet

High Line
New York City, New York

EXPLORE

Joaquin owns a landscaping company. He uses landscaping timbers to design flowerbeds and accent other features. Two timbers, with which he has recently worked, have lengths of $(x + 5)$ yards and $(2x - 6)$ yards. Let $f(x) = x + 5$ and $g(x) = 2x - 6$ represent the length of each timber. The timber lengths are always positive; however, the functions $f(x)$ and $g(x)$ model certain timber lengths, even though the functions have domains of all real numbers, including negative quantities.

1. If Joaquin joins the two timbers end to end, then the combined timbers have lengths that are added together. Complete a table like the one shown to compare the function values of $f(x)$, $g(x)$, and $(f + g)(x)$, the sum of $f(x)$ and $g(x)$.

TEKS

AR.3F Compare and contrast a function and possible functions that can be used to build it tabularly, graphically, and symbolically such as a quadratic function that results from multiplying two linear functions.

AR.4B Compare and contrast the results when adding two linear functions and multiplying two linear functions that are represented tabularly, graphically, and symbolically.

MATHEMATICAL PROCESS SPOTLIGHT

AR.1E Create and use representations to organize, record, and communicate mathematical ideas.

ELPS

4F Use visual and contextual support and support from peers and teachers to read grade-appropriate content area text, enhance and confirm understanding, and develop vocabulary, grasp of language structures, and background knowledge needed to comprehend increasingly challenging language.

VOCABULARY

domain, range, *x*-intercept, root, zero, *y*-intercept, degree

MATERIALS

- graphing technology

INTEGRATING TECHNOLOGY

Use the function editor of a graphing calculator or app to show $Y1 = f(x)$, $Y2 = g(x)$, and $Y3 = Y1 + Y2$. Use the table feature to generate a table and answer the questions that follow.

x	$f(x) = x + 5$	$g(x) = 2x - 6$	$(f + g)(x)$
−6	***−1***	***−18***	***−19***
−5	***0***	***−16***	***−16***
−4	***1***	***−14***	***−13***
−3	***2***	***−12***	***−10***
−2	***3***	***−10***	***−7***
−1	***4***	***−8***	***−4***
0	***5***	***−6***	***−1***
1	***6***	***−4***	***2***
2	***7***	***−2***	***5***
3	***8***	***0***	***8***
4	***9***	***2***	***11***

2. Use finite differences to classify and write a function rule for $(f + g)(x)$.
(f + g)(x) is a linear function and (f + g)(x) = 3x – 1.

3. Use the table to identify the slopes and intercepts of $f(x)$, $g(x)$, and $(f + g)(x)$.
See margin.

4. Compare and contrast the slopes and y-intercepts of $f(x)$, $g(x)$, and $(f + g)(x)$. How are they related?
See margin.

5. Use the function rules to graph $f(x)$, $g(x)$, and $(f + g)(x)$, the sum of $f(x)$ and $g(x)$, on the same coordinate plane.
See margin.

6. What relationships do you see among the domain and range of $f(x)$, $g(x)$, and $(f + g)(x)$ from the graphs?
Since all three functions are linear, all three functions have the same domain (all real numbers) and the same range (all real numbers).

7. Use the graph to compare and contrast the intercepts of $f(x)$, $g(x)$, and $(f + g)(x)$. What relationships or patterns do you notice?
See margin.

8. Compare and contrast the symbolic function rules for $f(x)$, $g(x)$, and $(f + g)(x)$. How are they related?
See margin.

9. Joaquin uses the two sizes of timbers to outline a rectangular flowerbed, one timber as the width and the other timber as the length. The area of the flowerbed is the product of the length and width. Add a column to your table for values compare the function values of $f(x)$, $g(x)$, and $(f \cdot g)(x)$, the product of $f(x)$ and $g(x)$.

3. *$f(x)$: slope = 1,*
x-intercept (-5, 0),
y-intercept (0, 5)

$g(x)$: slope = 2,
x-intercept (3, 0),
y-intercept (0, -6)

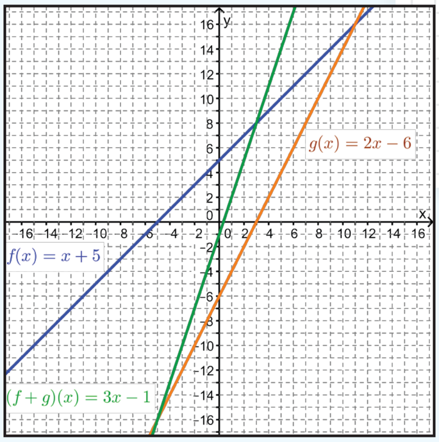
$(f + g)(x)$: slope = 3,
x-intercept ($\frac{1}{3}$, 0),
y-intercept (0, -1)

4. *Answers may vary. Possible response:*

The slope of $(f + g)(x)$ is the sum of the slopes of $f(x)$ and $g(x)$.

The y-coordinate of the y-intercept of $(f + g)(x)$ is the sum of the y-coordinates of the y-intercepts of $f(x)$ and $g(x)$.

5.



7. *Answers may vary. Possible responses:*

- *The x-intercept of $f(x)$ has the same x-coordinate as the intersection point of $g(x)$ and $(f + g)(x)$.*
- *The x-intercept of $g(x)$ has the same x-coordinate as the intersection point of $f(x)$ and $(f + g)(x)$.*
- *The y-coordinate of the y-intercept of $(f + g)(x)$ is the sum of the y-coordinates of the y-intercepts of $f(x)$ and $g(x)$.*

8. *Answers may vary. Possible responses:*

If you write each function rule in $y = mx + b$ form, then:

- *For $(f + g)(x)$, the value of m is the sum of the values of m for $f(x)$ and $g(x)$.*
- *For $(f + g)(x)$, the value of b is the sum of the values of b for $f(x)$ and $g(x)$.*

x	$f(x) = x + 5$	$g(x) = 2x - 6$	$(f + g)(x)$	$(f \cdot g)(x)$
−6	**−1**	**−18**	**−19**	**18**
−5	**0**	**−16**	**−16**	**0**
−4	**1**	**−14**	**−13**	**−14**
−3	**2**	**−12**	**−10**	**−24**
−2	**3**	**−10**	**−7**	**−30**
−1	**4**	**−8**	**−4**	**−32**
0	**5**	**−6**	**−1**	**−30**
1	**6**	**−4**	**2**	**−24**
2	**7**	**−2**	**5**	**−14**
3	**8**	**0**	**8**	**0**
4	**9**	**2**	**11**	**18**

10. Use finite differences to classify and write a function rule for $(f \cdot g)(x)$.
$(f \cdot g)(x)$ is a quadratic function and $(f \cdot g)(x) = (x + 5)(2x - 6) = 2x^2 + 4x - 30$.

11. Use the table to identify the intercepts of $f(x)$, $g(x)$, and $(f \cdot g)(x)$.
See margin.

12. Compare and contrast the intercepts of $f(x)$, $g(x)$, and $(f \cdot g)(x)$. How are they related?
See margin.

13. Use the function rules to graph $f(x)$, $g(x)$, and $(f \cdot g)(x)$, the product of $f(x)$ and $g(x)$, on the same coordinate plane.
See margin.

14. When the function values for $(f \cdot g)(x)$ are greater than 0, what do you notice about the sign of the function values for both $f(x)$ and $g(x)$?
The function values for both f(x) and g(x) are either both positive ($x > 3$) or both negative ($x < -5$).

15. When the function values for $(f \cdot g)(x)$ are less than 0, what do you notice about the sign of the function values for both $f(x)$ and $g(x)$?
The function values for f(x) are positive and the function values for g(x) are negative.

16. Use the graph to compare and contrast the intercepts of $f(x)$, $g(x)$, and $(f \cdot g)(x)$. What relationships or patterns do you notice?
See margin.

17. Compare and contrast the symbolic function rules for $f(x)$, $g(x)$, and $(f \cdot g)(x)$. How are they related?
See margin.

11. *f(x): x-intercept (-5, 0), y-intercept (0, 5)*

g(x): x-intercept (3, 0), y-intercept (0, -6)

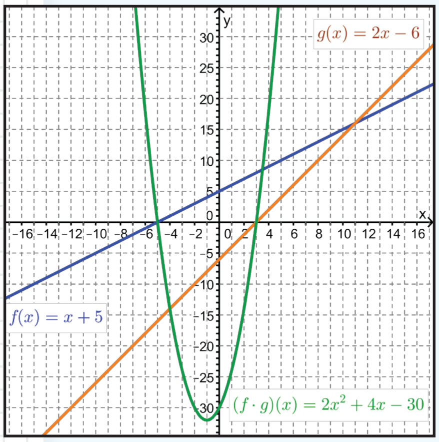
$(f \cdot g)(x)$: x-intercepts (-5, 0) and (3, 0), y-intercept (0, -30)

12. *Answers may vary. Possible response:*

The x-intercepts of $(f \cdot g)(x)$ are the same as the x-intercepts of both f(x) and g(x).

The y-coordinate of the y-intercept of $(f \cdot g)(x)$ is the product of the y-coordinates of the y-intercepts of f(x) and g(x).

13.



16. *Answers may vary. Possible responses:*

The x-intercepts of $(f \cdot g)(x)$ are the same as the x-intercepts of both f(x) and g(x).

The y-coordinate of the y-intercept of $(f \cdot g)(x)$ is the product of the y-coordinates of the y-intercepts of f(x) and g(x).

17. *Answers may vary. Possible response:*

The function rule for $(f \cdot g)(x)$ in factored form is the same as f(x) times g(x).

The functions f(x) or g(x) are factors of $(f \cdot g)(x)$, but are not always the only factors of $(f \cdot g)(x)$.

REFLECT ANSWERS:

A table of function values shows how the slopes and intercepts of two linear function addends and their sum are related. A graph shows how intercepts are related to intersection points and other graphical features. The symbolic function rule reveals how the slopes and y-coordinates of the y-intercepts are related.

A table of values shows how the intercepts of the two linear factors are related to the intercepts of the product. A graph shows how the x-intercepts and y-intercepts of the linear factors are related to the intercepts of the quadratic product. The symbolic function rule for the quadratic product function, when factored, shows the connections to the linear factors.

The sum of two linear functions is a linear function but the product of two linear functions is a quadratic function. Multiplying two polynomial functions increases the degree of the function due to the laws of exponents. A function of degree one multiplied by a function of degree one generates a function of degree two.

The y-coordinates of the y-intercepts of two linear functions are added together to generate the y-intercept of the sum function, just as the y-coordinates of the y-intercepts of two linear functions are multiplied to generate the y-intercept of the product function.

The x-intercepts of two linear addends are not related to the x-intercept of the sum, but the x-intercepts of two linear factors each become an x-intercept of the quadratic product.

REFLECT

- **How do the different representations (table, graph, or symbolic function rule) reveal different properties of two linear function addends and their sum?**
  See margin.
- **How do the different representations (table, graph, or symbolic function rule) reveal different properties of two linear function factors and their product?**
  See margin.
- **Compare and contrast the results of adding two linear functions with the results of multiplying two linear functions.**
  See margin.

EXPLAIN

Any two functions can be combined using addition or multiplication. Sums and products of polynomial functions have special patterns. In particular, there are important relationships that emerge when adding two linear functions or multiplying two linear functions. In his role with the landscaping company, Joaquin had landscaping timbers that were of two different lengths: $x + 5$ yards and $2x - 6$ yards. Joaquin has other types of landscaping timbers that have lengths that can be represented using linear functions.

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ADDING LINEAR FUNCTIONS

Any linear function can be written in a general form, $f(x) = ax + b$. In this form, which is very similar to slope-intercept form, $y = mx + b$, a represents the slope or rate of change and b represents the y-coordinate of the y-intercept.

Suppose that one type of timber has a length $p(x) = ax + b$ and another timber has a length $q(x) = cx + d$.

If Joaquin lays the two timbers end to end, then he can combine their lengths using addition of polynomials.

If $k(x)$ represents the combined length of the two timbers, then $k(x) = p(x) + q(x)$.

$k(x) = p(x) + q(x)$
$k(x) = (ax + b) + (cx + d)$
$k(x) = ax + b + cx + d$
$k(x) = ax + cx + b + d$
$k(x) = (a + c)x + (b + d)$

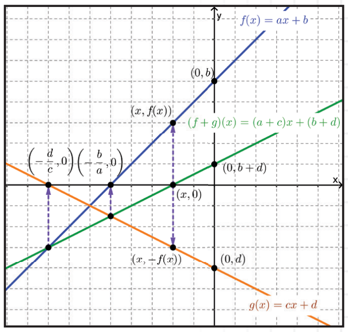
FOR $k(x)$

- The slope or rate of change is the sum of the slopes of the two linear addends, $(a + c)$.
- The y-coordinate of the y-intercept is the sum of the y-coordinates of the y-intercepts of the two linear addends, $(0, b + d)$.

Symbolically, you can verify that the sum of two linear functions will always be a linear function.

In a graph, you can visually observe the relationships among the x-intercepts and y-intercepts of two linear addends, in general, $f(x)$ and $g(x)$, and the linear sum, in general, $(f + g)(x)$.

- The x-intercept of $f(x)$ has the same x-coordinate as the intersection of $g(x)$ and the sum, $(f + g)(x)$.
- The x-intercept of $g(x)$ has the same x-coordinate as the intersection of $f(x)$ and the sum, $(f + g)(x)$.
- The x-intercept of $(f + g)(x)$ has the same x-coordinate as the input value generating $f(x) = -g(x)$.
- The y-intercept of $(f + g)(x)$ has a y-coordinate that is the sum of the y-coordinates of the y-intercepts of $f(x)$ and $g(x)$.



In a table, you can also observe relationships among the intercepts of the two addends and the sum.

x	$f(x) = x + 5$	$g(x) = 2x - 2$	$(f + g)(x)$
−6	−1	−14	−15
−5	0	−12	−12
−4	1	−10	−9
−3	2	−8	−6
−2	3	−6	−3
−1	4	−4	0
0	5	−2	3
1	6	0	6
2	7	2	9
3	8	4	12
4	9	6	15

At the x-intercept of one addend, the other addend and the sum have the same function value, so the other addend and sum function intersect.

At the x-intercept of the sum function, the function values for each addend are opposites and have a sum of 0 (i.e., $f(x) = -g(x)$).

The y-intercept of the sum function has a y-coordinate that is the sum of the y-coordinates of the y-intercepts of both addends.

At the x-intercept of one addend, the other addend and the sum have the same function value, so the other addend and sum function intersect.

INSTRUCTIONAL HINT

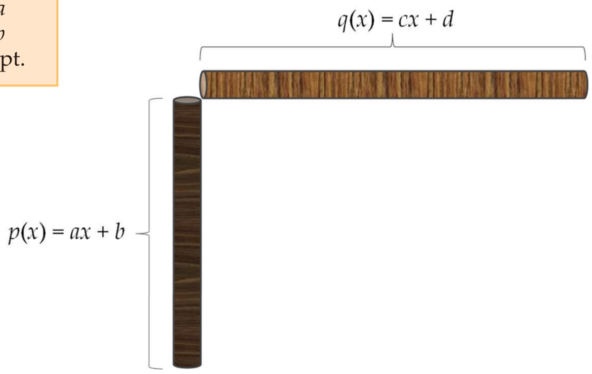
To help students understand the relationships presented here, have students turn to a partner and use mathematical terms and their own words to explain the relationships to each other.

MULTIPLYING LINEAR FUNCTIONS

Let's take a different look at the two landscaping timbers with lengths $p(x) = ax + b$ and $q(x) = cx + d$.

Any linear function can be written in general form, $f(x) = ax + b$. In this form, which is very similar to slope-intercept form, $y = mx + b$, a represents the slope or rate of change and b represents the y-coordinate of the y-intercept.

If Joaquin sets the two timbers perpendicular at one end, then he can frame out a rectangular shape. The area of the rectangle is the product of the length and width of the rectangle, which can be calculated using multiplication of polynomials.



If $A(x)$ represents the area of the region framed by the two timbers, then $A(x) = p(x) \cdot q(x)$.

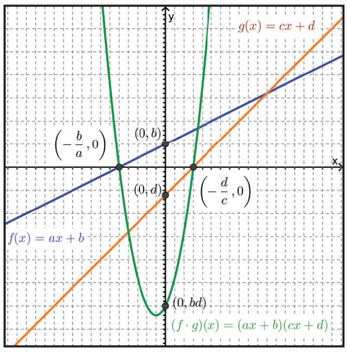
$$A(x) = p(x) \cdot q(x)$$
$$A(x) = (ax + b) \cdot (cx + d)$$
$$A(x) = acx^2 + axd + bcx + bd$$
$$A(x) = (ac)x^2 + (ad + bc)x + bd$$

FOR *A*(*x*)

- The product of two linear functions is a quadratic function.
- The y-coordinate of the y-intercept is the product of the y-coordinates of the y-intercepts of the two linear factors, $(0, bd)$.

In a graph, you can visually observe the relationships among the x-intercepts and y-intercepts of two linear factors, in general, $f(x)$ and $g(x)$, and the quadratic product, in general, $(f \cdot g)(x)$.

- The x-intercepts of $f(x)$ and $g(x)$ are each the same as an x-intercept of the product, $(f \cdot g)(x)$. As long as $f(x)$ is not equivalent to $g(x)$, there will be two unique x-intercepts of $(f \cdot g)(x)$. If $f(x)$ is equivalent to $g(x)$, then there will be one x-intercept of $(f \cdot g)(x)$ at the vertex of $(f \cdot g)(x)$.
- The y-intercept of $(f \cdot g)(x)$ has a y-coordinate that is the product of the y-coordinates of the y-intercepts of $f(x)$ and $g(x)$.



INSTRUCTIONAL HINT

To help students understand the relationships presented here, have students turn to a different partner than the one they spoke to on page 529 and use mathematical terms and their own words to explain the relationships to each other.

In a table, you can also observe relationships among the intercepts of the two linear function factors and the quadratic function product.

x	$f(x) = x + 5$	$g(x) = 2x - 2$	$(f \cdot g)(x)$
−6	−1	−14	14
−5	0	−12	0
−4	1	−10	−10
−3	2	−8	−16
−2	3	−6	−18
−1	4	−4	−16
0	5	−2	−10
1	6	0	0
2	7	2	14
3	8	4	32
4	9	6	54

The x-intercept of one linear factor is the same as an x-intercept of the quadratic product.

The y-intercept of the product function has a y-coordinate that is the product of the y-coordinates of the y-intercepts of both factors.

The x-intercept of one linear factor is the same as an x-intercept of the quadratic product.

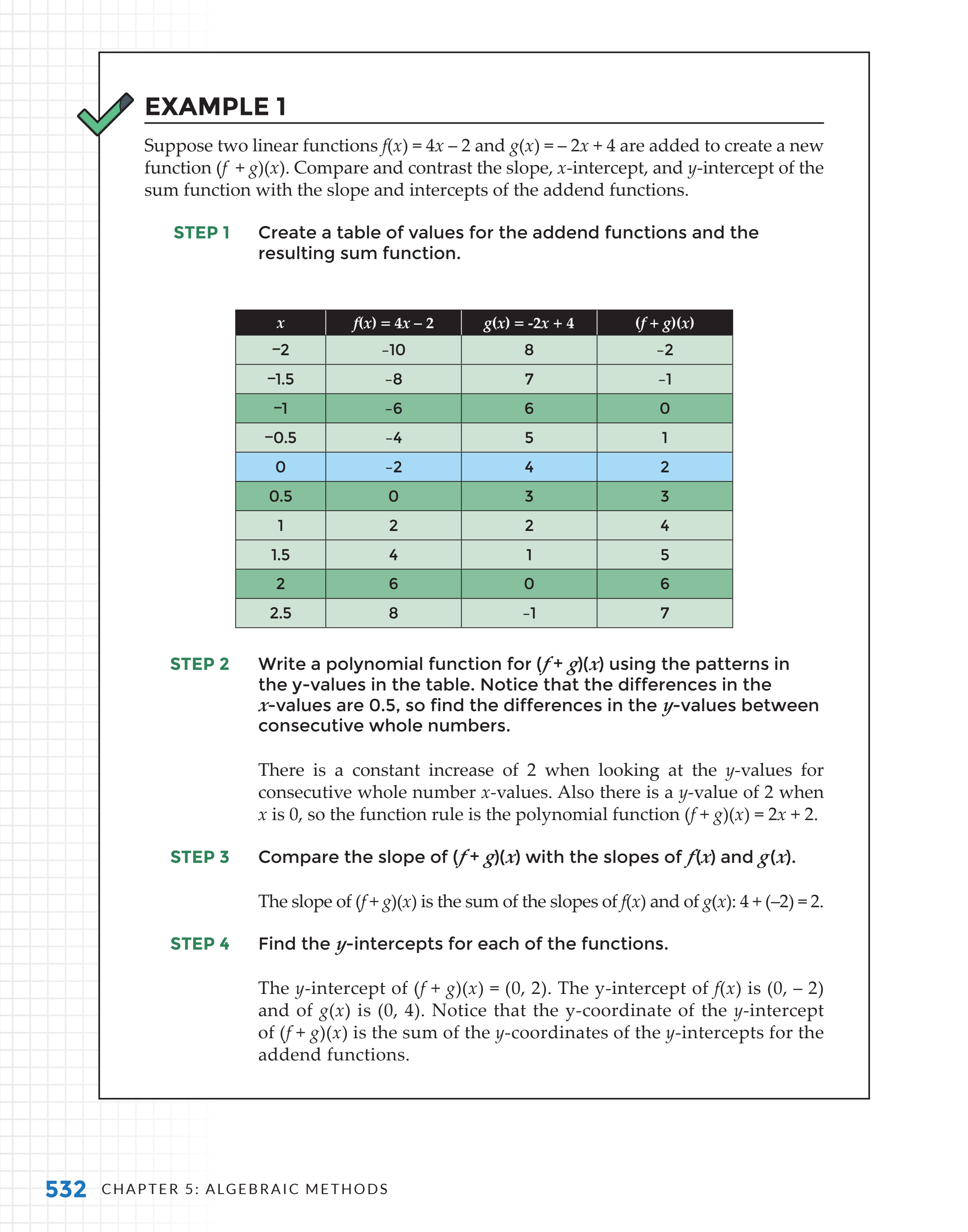
ADDING AND MULTIPLYING LINEAR FUNCTIONS

When adding two linear functions, the function representing the sum is a linear function.

- The y-intercept of the sum function is the sum of the y-intercepts of the two addend functions.
- The x-intercept of one addend function has the same x-coordinate as the intersection of the second addend function and the sum function.
- The slope of the sum function is the sum of the slopes of the two addend functions.

When multiplying two linear functions, the function representing the product is a quadratic function.

- The y-intercept of the product function is the product of the y-intercepts of the two factor functions.
- The x-intercepts of each linear function factor are the same as each x-intercept of the quadratic function product.
- If the quadratic function is written in factored form instead of polynomial form, each factor is equivalent to one of the linear function factors used to generate the quadratic function product.



EXAMPLE 1

Suppose two linear functions $f(x) = 4x - 2$ and $g(x) = -2x + 4$ are added to create a new function $(f + g)(x)$. Compare and contrast the slope, x-intercept, and y-intercept of the sum function with the slope and intercepts of the addend functions.

STEP 1 **Create a table of values for the addend functions and the resulting sum function.**

x	$f(x) = 4x - 2$	$g(x) = -2x + 4$	$(f + g)(x)$
−2	-10	8	-2
−1.5	-8	7	-1
−1	-6	6	0
−0.5	-4	5	1
0	-2	4	2
0.5	0	3	3
1	2	2	4
1.5	4	1	5
2	6	0	6
2.5	8	-1	7

STEP 2 **Write a polynomial function for $(f + g)(x)$ using the patterns in the y-values in the table. Notice that the differences in the x-values are 0.5, so find the differences in the y-values between consecutive whole numbers.**

There is a constant increase of 2 when looking at the y-values for consecutive whole number x-values. Also there is a y-value of 2 when x is 0, so the function rule is the polynomial function $(f + g)(x) = 2x + 2$.

STEP 3 **Compare the slope of $(f + g)(x)$ with the slopes of $f(x)$ and $g(x)$.**

The slope of $(f + g)(x)$ is the sum of the slopes of $f(x)$ and of $g(x)$: $4 + (-2) = 2$.

STEP 4 **Find the y-intercepts for each of the functions.**

The y-intercept of $(f + g)(x) = (0, 2)$. The y-intercept of $f(x)$ is $(0, -2)$ and of $g(x)$ is $(0, 4)$. Notice that the y-coordinate of the y-intercept of $(f + g)(x)$ is the sum of the y-coordinates of the y-intercepts for the addend functions.

ADDITIONAL EXAMPLES

Given the linear functions $f(x)$ and $g(x)$, write a polynomial for their sum $(f + g)(x)$. Use a table or graph to compare and contrast their critical attributes (slope, x-intercepts, y-intercepts, and points of intersection).

1. $f(x) = \frac{1}{2}x + 5$ and $g(x) = -\frac{3}{2}x - 6$

$(f + g)(x) = -x - 1$

Slopes: Slope of $f(x) = \frac{1}{2}$, slope of $g(x) = -\frac{3}{2}$, and slope of $f + gx = -1$. The slope of $(f + g)(x)$ is the sum of the slopes of the addend functions.

y-intercepts: $f(x)$: (0, 5); $g(x)$: (0, -6); $(f + g)(x)$: (0, -1). The y-intercept of $(f + g)(x)$ is the sum of the y-coordinates of the y-intercepts of the addends.

x-intercepts: $f(x)$: (-10, 0); $g(x)$: (-4, 0); $(f + g)(x)$: (-1, 0). The intersection of $f(x)$ and $(f + g)(x)$ is (-4, 3) and has the same x-coordinate as the x-coordinate of the x-intercept of $g(x)$. Also, the intersection of $g(x)$ and $(f + g)(x)$ is (-10, 9) and has the same x-coordinate as the x-coordinate of the x-intercept of $f(x)$.

2. $f(x) = 7x - 14$ and $g(x) = -12x + 6$

$(f + g)(x) = -5x - 8$

Slopes: Slope of $f(x) = 7$, slope of $g(x) = -12$, and slope of $(f + g)(x) = -5$. The slope of $(f + g)(x)$ is the sum of the slopes of the addend functions.

y-intercepts: $f(x)$: (0, -14); $g(x)$: (0, 6); $(f + g)(x)$: (0, -8). The y-intercept of $(f + g)(x)$ is the sum of the y-coordinates of the y-intercepts of the addends.

x-intercepts: $f(x)$: (2, 0); $g(x)$: $(\frac{1}{2}, 0)$; $(f + g)(x)$: $(\frac{8}{5}, 0)$. The intersection of $f(x)$ and $(f + g)(x)$ is $(\frac{1}{2}, \frac{21}{2})$ and has the same x-coordinate as the x-coordinate of the x-intercept of $g(x)$. Also, the intersection of $g(x)$ and $(f + g)(x)$ is (2, -18) and has the same x-coordinate as the x-coordinate of the x-intercept of $f(x)$.

STEP 5 **Find the x-intercepts for each of the functions.**

For $(f + g)(x)$, the x-intercept is (-1, 0). Notice that for $x = -1$, the y-coordinates for the addend functions are opposites, -6 and 6, and so their sum is the y-coordinate for the x-intercept for $(f + g)(x)$.

For $f(x)$, the x-intercept is (0.5, 0). The y-coordinates of $(f + g)(x)$ and $g(x)$ are both 3 for $x = 0.5$. So the graphs of $(f + g)(x)$ and $g(x)$ would intersect at this point.

For $g(x)$, the x-intercept is (2, 0). The y-coordinates of $(f + g)(x)$ and $f(x)$ are both 6 for $x = 2$. So the graphs of $(f + g)(x)$ and $f(x)$ would intersect at this point.

When either $f(x)$ or $g(x)$ has an x-intercept, the other function intersects with $(f + g)(x)$. An intersection point means that the function values are equal. Symbolically, you can see this relationship.
$f(x) + g(x) = (f + g)(x)$, so

- When $g(x)$ has an x-intercept, $g(x) = 0$, so $(f + g)(x) = f(x) + 0$ and $(f + g)(x) - f(x) = g(x)$.
- When $f(x)$ has an x-intercept, $f(x) = 0$, so $(f + g)(x) = g(x) + 0$ and $(f + g)(x) - g(x) = f(x)$.

YOU TRY IT! #1

Given the linear functions $m(x) = x - 3$ and $n(x) = 3 + x$, write a polynomial for their sum $(m + n)(x)$, and graph all three functions on a coordinate grid. Using the graphs, compare and contrast their critical attributes (slope, x-intercepts, y-intercepts, and points of intersection).

See margin.

EXAMPLE 2

Generate a table of values and a graph to represent $f(x) = 2x - 3$ and $g(x) = -4x + 2$. Then generate values for $(f \cdot g)(x) = g(x) \cdot f(x)$ to write a polynomial function, and graph it in the same coordinate plane. Use the equations, tables, and graphs to compare $(f \cdot g)(x)$ with $f(x)$ and $g(x)$.

YOU TRY IT! #1 ANSWER:

$(m + n)(x) = (x - 3) + (3 + x) = 2x$

Slopes:

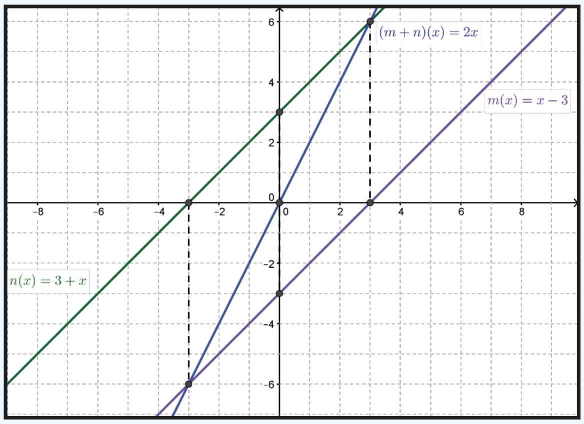
- *Slope of $m(x) = 1$, slope of $n(x) = 1$, and slope of $(m + n)(x) = 2$*
- *The slope of $(m + n)(x)$ is 2, the sum of the slopes of the addend functions. Because the slopes of $m(x)$ and $n(x)$ are the same and their y-intercepts are different, their graphs are parallel and do not intersect.*

y-intercepts:

- *$m(x)$: (0, –3); $n(x)$: (0, 3); $(m + n)(x)$: (0, 0)*
- *The y-intercept of $(m + n)(x)$ is the sum of the y-coordinates of the y-intercepts of the addends.*

x-intercepts:

- *$m(x)$: (3, 0); $n(x)$: (–3, 0); $(m + n)(x)$: (0, 0)*
- *The intersection of $m(x)$ and $(m + n)(x)$ is (– 3, – 6) and has the same x-coordinate as the x-coordinate of the x-intercept of $n(x)$. Also, the intersection of $n(x)$ and $(m + n)(x)$, which is (3, 6), has the same x-coordinate as the x-coordinate of the x-intercept of $m(x)$.*

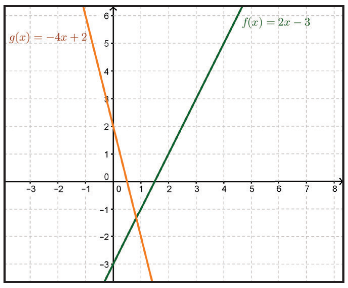


STEP 1 **Generate a table of values for the two linear functions.**

$\Delta x = 1 - 0 = 1$
$\Delta x = 2 - 1 = 1$
$\Delta x = 3 - 2 = 1$
$\Delta x = 4 - 3 = 1$

x	$f(x) = 2x - 3$	$g(x) = -4x + 2$
-1	2(-1) – 3 = -5	-4(-1) + 2 = 6
0	2(0) – 3 = -3	-4(0) + 2 = 2
1	2(1) – 3 = -1	-4(1) + 2 = -2
2	2(2) – 3 = 1	-4(2) + 2 = -6
3	2(3) – 3 = 3	-4(3) + 2 = -10

STEP 2 **Notice that the constant first difference in the function values for $f(x)$ is 2 and for $g(x)$, is -4. This coincides with the slopes seen as a in their equations. Graph $f(x)$ and $g(x)$ using the generated values. Determine their x-intercepts and y-intercepts using the table of values and the properties of algebra.**



The x-coordinate for y-intercepts is zero. Using the table, the y-intercept for $f(x)$ is $(0, -3)$ and the y-intercept for $g(x)$ is $(0, 2)$. These coincide with b in their equations.

The y-value for x-intercepts is zero. Using algebraic properties, solve for x for each of the functions.

$$f(x) = 2x - 3 \quad 0 = 2x - 3 \quad 3 = 2x \quad \frac{3}{2} = x$$

$$g(x) = -4x + 2 \quad 0 = -4x + 2 \quad -2 = -4x \quad \frac{-2}{-4} = \frac{1}{2} = x$$

So the x-intercept for $f(x)$ is $(\frac{3}{2}, 0)$ and for $g(x)$ is $(\frac{1}{2}, 0)$. These points are verified in the graph.

STEP 3 **Generate a table of values for $(f \cdot g)(x) = g(x) \cdot f(x)$. Determine what type of polynomial function best represents $(f \cdot g)(x)$ and write a function rule, using the patterns in the table and also symbolically.**

x	$f(x)$	$g(x)$	$(f \cdot g)(x)$
-1	-5	6	-30
0	-3	2	-6
1	-1	-2	2
2	1	-6	-6
3	3	-10	-30
4	5	-14	-70

24
8
-8
-24
-40

-16
-16
-16
-16

The second finite differences are constant, so the function (f · g)(x) is quadratic.

$2a = -16$	$a + b = 8$	$c = -6$
$a = -8$	$-8 + b = 8$	
	$-8 + b + 8 = 8 + 8$	
	$b = 16$	

The polynomial function is $(f \cdot g)(x) = -8x^2 + 16x - 6$.

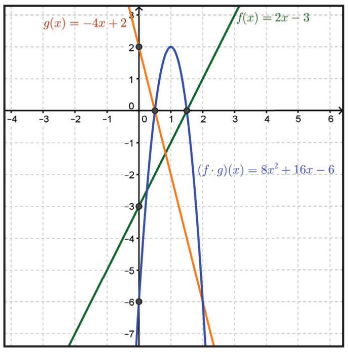
The polynomials for the factor functions can also be multiplied to write the polynomial product.

$$(f \cdot g)(x) = (2x - 3)(-4x + 2)$$
$$(f \cdot g)(x) = 2x(-4x + 2) - 3(-4x + 2)$$
$$(f \cdot g)(x) = -8x^2 + 4x + 12x - 6$$
$$(f \cdot g)(x) = -8x^2 + 16x - 6$$

Using either method, the same function rule can be written for the product.

STEP 4 **Graph the product $(f \cdot g)(x)$ on the same coordinate grid with the factors. Identify the intercepts of $(f \cdot g)(x)$ and compare them with those of the factors.**

	x-intercept	y-intercept
$f(x)$	$(\frac{3}{2}, 0)$	(0, -3)
$g(x)$	$(\frac{1}{2}, 0)$	(0, 2)
$(f \cdot g)(x)$	$(\frac{1}{2}, 0)$ and $(\frac{3}{2}, 0)$	(0, -6)



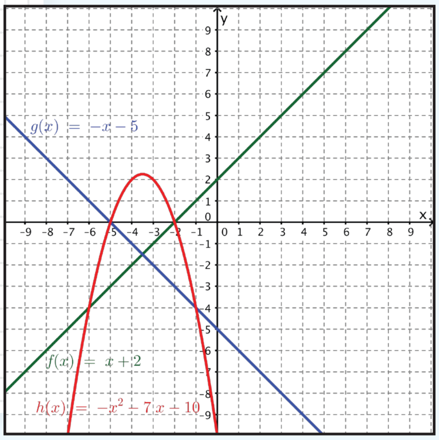
The x-intercepts of the product function are the same as the x-intercept of each of the factor functions. The y-coordinate of the y-intercept of the product function is the product of the y-coordinates of the y-intercepts of the two factor functions.

ADDITIONAL EXAMPLE

Given the linear functions $f(x) = x + 2$ and $g(x) = -x - 5$, generate a table of values and a graph. Then generate values for $(f \cdot g)(x) = f(x) \cdot g(x)$ to write a polynomial function, and graph it in the same coordinate plane. Use the equations, tables, and graphs to compare $(f \cdot g)(x)$ with $f(x)$ and $g(x)$.

x	$f(x)$	$g(x)$	$(f \cdot g)(x)$
-6	-4	1	-4
-5	-3	0	0
-4	-2	-1	2
-3	-1	-2	2
-2	0	-3	0
-1	1	-4	-4
0	2	-5	-10

The product $f(x) \cdot g(x)$ produces the polynomial function $(f \cdot g)(x) = (x + 2)(-x - 5) = -x^2 - 7x - 10$.

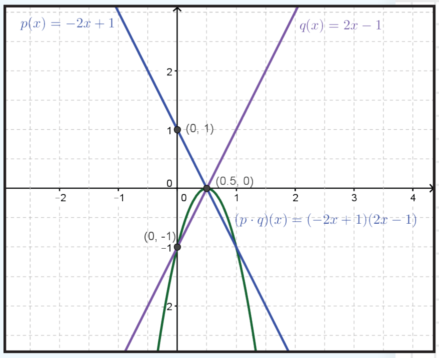


	x-INTERCEPT	y-INTERCEPT
$f(x)$	(-2, 0)	(0, 2)
$g(x)$	(-5, 0)	(0, -5)
$(f \cdot g)(x)$	(-2, 0) AND (-5, 0)	(0, -10)

The x-intercepts of the product function are the same as the x-intercept of each of the factor functions. The y-coordinate of the y-intercept of the product function is the product of the y-coordinates of the y-intercepts of the two factor functions.

YOU TRY IT! #2 ANSWER:

The graphs of all three functions intersect at (0.5, 0), which means that $(p \cdot q)(x)$ has two identical x-intercepts. The y-intercept of $p(x)$ is (0, 1) and of $q(x)$ is (0, – 1), which is seen as b in their equations. And so the y-coordinate of the y-intercept of $(p \cdot q)(x)$ is the product of the y-coordinates of the y-intercepts of the factors. The product $p(x) \cdot q(x)$ produces the polynomial function $(p \cdot q)(x) = (-2x + 1)(2x - 1) = -4x^2 + 4x - 1$.



YOU TRY IT! #2

Complete the table of values to graph $p(x) = -2x + 1$ and $q(x) = 2x - 1$. Then generate values for $(p \cdot q)(x) = p(x) \cdot q(x)$ to graph the product function in the same coordinate plane. Use the polynomials for the factor functions to write a polynomial for the product function.

x	$p(x)$	$q(x)$	$p(x) \cdot q(x)$
-0.5	*2*	*-2*	*-4*
0	*1*	*-1*	*-1*
0.5	*0*	*0*	*0*
1	*-1*	*1*	*-1*
1.5	*-2*	*2*	*-4*

See margin.

EXAMPLE 3

For $f(x) = 2 + 2x$ and $g(x) = -3 - x$, generate a table of values for each function and write polynomials to represent their sum $(f + g)(x)$ and their product $(f \cdot g)(x)$. Compare their x-intercepts, y-intercepts and slopes, where appropriate. Graph the coordinate pairs on the same grid to verify these attributes.

STEP 1 Complete the table values for each of the functions.

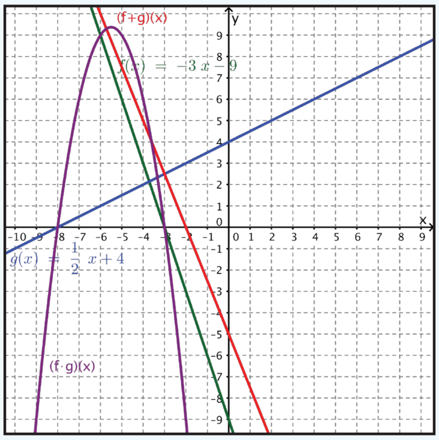
x	$f(x) = 2 + 2x$	$g(x) = -3 - x$	$f(x) + g(x)$	$f(x) \cdot g(x)$
-4	-6	1	-5	-6
-3	-4	0	-4	0
-2	-2	-1	-3	2
-1	0	-2	-2	0
0	2	-3	-1	-6
1	4	-4	0	-16
2	6	-5	1	-30

ADDITIONAL EXAMPLE

For $f(x) = -3x - 9$ and $g(x) = \frac{1}{2}x + 4$, generate a table of values for each function, and then write polynomials to represent their sum $(f + g)(x)$ and their product $(f \cdot g)(x)$. Compare their x-intercepts, y-intercepts, and slopes, where appropriate. Graph the coordinate pairs on the same grid to verify these attributes.

The following table, graph, and chart show the polynomial representations of the functions, x-intercepts, y-intercepts, and slopes, where appropriate.

x	$f(x) = -3x - 9$	$g(x) = \frac{1}{2}x + 4$	$(f + g)(x)$	$(f \cdot g)(x)$
-8	15	0	15	0
-7	12	0.5	12.5	6
-6	9	1	1	9
-5	6	1.5	7.5	9
-4	3	2	5	6
-3	0	2.5	2.5	0
-2	-3	3	0	-9
-1	-6	3.5	-2.5	-21
0	-9	4	-5	-36



Answer continued on bottom of page 537.

STEP 2 **The first differences in the table values for the sum function are all 1, so it is represented by the polynomial function $(f + g)(x) = x - 1$ where $a = 1$ and $b = -1$. A polynomial representing the sum can also be written by adding the polynomials representing $f(x)$ and $g(x)$ and simplifying.**

$(f + g)(x) = f(x) + g(x)$
$(f + g)(x) = (2 + 2x) + (-3 – x)$
$(f + g)(x) = 2x – x + 2 – 3$
$(f + g)(x) = x – 1$

For $(f \cdot g)(x)$, multiply the factor functions and simplify the resulting polynomial.

$(f \cdot g)(x) = f(x) \cdot g(x)$
$(f \cdot g)(x) = (2 + 2x)(-3 – x)$
$(f \cdot g)(x) = 2(-3 – x) + 2x(-3 – x)$
$(f \cdot g)(x) = -6 – 2x – 6x – 2x^2$
$(f \cdot g)(x) = -2x^2 – 8x – 6$

Verify that the polynomial represents the values in the table. For example, $(f \cdot g)(0) = -2(0)^2 – 8(0) – 6 = -6$ and $(f \cdot g)(2) = -2(2)^2 – 8(2) – 6 = -30$.

STEP 3 **Use the tables to identify the slopes and intercepts of $f(x)$, $g(x)$, and $(f + g)(x)$.**

$f(x)$: slope = 2, x-intercept (−1, 0), y-intercept (0, 2)
$g(x)$: slope = −1, x-intercept (−3, 0), y-intercept (0, −3)
$(f + g)(x)$: slope = 1, x-intercept (1, 0), y-intercept (0, −1)

The slope of the sum function is the sum of the slopes of the addends, 2 + (−1) = 1.
The x-intercepts are not related.
The y-coordinates of the y-intercept of the sum function is the sum of the y-coordinates of the y-intercepts of the addends, 2 + (−3) = – 1.

STEP 4 **Use the tables to identify the slopes and intercepts of $f(x)$, $g(x)$, and $(f \cdot g)(x)$.**

$f(x)$: x-intercept (−1, 0), y-intercept (0, 2)
$g(x)$: x-intercept (−3, 0), y-intercept (0, −3)
$(f \cdot g)(x)$: x-intercept (−3, 0) and (−1, 0), y-intercept (0, −6)

The product of two linear functions results in a quadratic function, so slope is not considered. The x-intercepts of the product function are the x-intercepts of each of the factors. The y-coordinate of the y-intercept of the product function is the product of the y-coordinates of the y-intercepts of the factors, (2)(-3) = -6.

	POLYNOMIAL FUNCTION	X-INTERCEPT	Y-INTERCEPT	SLOPE
$f(x)$	$-3x - 9$	(-3, 0)	(0, -9)	-3
$g(x)$	$\frac{1}{2}x + 4$	(-8, 0)	(0, 4)	$\frac{1}{2}$
$(f + g)(x)$	$-\frac{5}{2}x - 5$	(-2, 0)	(0, 5)	$\frac{-5}{2}$
$(f \cdot g)(x)$	$-\frac{3}{2}x^2 - 16.5x - 36$	(-3, 0) AND (-8, 0)	(0, -36)	N/A

YOU TRY IT! #3

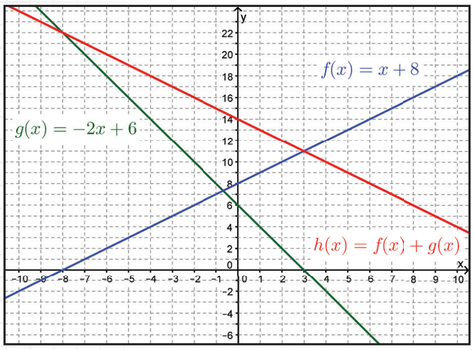
For $f(x) = 1.5x$ and $g(x) = -x + 1.5$, complete the table of values for each function, and then write polynomials to represent their sum $(f + g)(x)$ and their product $(f \cdot g)(x)$. Compare their x-intercepts, y-intercepts, and slopes, where appropriate. Graph the coordinate pairs on the same grid to verify these attributes.

x	$f(x) = 1.5x$	$g(x) = -x + 1.5$	$f(x) + g(x)$	$f(x) \cdot g(x)$
– 4				
– 3				
– 2				
– 1				
0				
1				
2				

See margin.

PRACTICE/HOMEWORK

1. The graph of $f(x) = x + 8$, $g(x) = -2x + 6$, and $h(x) = f(x) + g(x)$ is shown.



A. Both $f(x)$ and $g(x)$ are linear functions. What type of function is $h(x) = f(x) + g(x)$?
The function h(x) is a linear function.

B. What is the ordered pair of the point where $h(x)$ intersects $f(x)$? What is the ordered pair of the x-intercept of $g(x)$? What is the same in both ordered pairs?
See margin.

C. What is the ordered pair of the point where $h(x)$ intersects $g(x)$? What is the ordered pair of the x-intercept of $f(x)$? What is the same in both ordered pairs?
See margin.

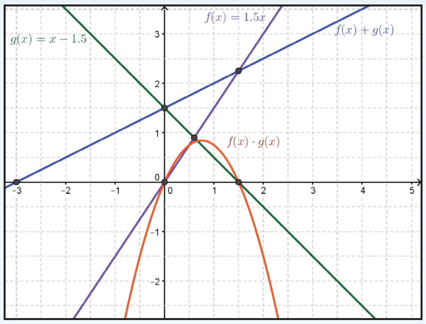
1B. *h(x) intersects f(x) at (3, 11). The x-intercept of g(x) is (3, 0). Both ordered pairs have an x-value of 3.*

1C. *h(x) intersects g(x) at (-8, 22). The x-intercept of f(x) is (-8, 0). Both ordered pairs have an x-value of -8.*

YOU TRY IT! #3 ANSWER:

The completed table, graph, and chart show the polynomial representations of the functions, the x-intercepts, y-intercepts, and slopes, where appropriate.

x	$f(x) = 1.5x$	$g(x) = -x + 1.5$	$f(x) + g(x)$	$f(x) \cdot g(x)$
– 4	-6	5.5	-0.5	-33
– 3	-4.5	4.5	0	-20.25
– 2	-3	3.5	0.5	-10.5
– 1	-1.5	2.5	1	-3.75
0	0	1.5	1.5	0
1	1.5	0.5	2	0.75
2	3	-0.5	2.5	-1.5



	POLYNOMIAL FUNCTION	x-INTERCEPT	y-INTERCEPT	SLOPE
$f(x)$	$1.5x$	(0, 0)	(0, 0)	1.5
$g(x)$	$-x + 1.5$	(1.5, 0)	(0, 1.5)	-1
$(f+g)(x)$	$0.5x + 1.5$	(-3, 0)	(0, 1.5)	0.5
$(f \cdot g)(x)$	$-1.5x^2 + 2.25x$	(0, 0) and (1.5, 0)	(0, 0)	N/A

Use the following information to answer questions 2 – 6.

Two linear functions, $f(x) = 3x - 6$ and $g(x) = x + 2$, are added to create a new function $(f + g)(x)$.

2. Complete the table of values for the addend functions and the resulting sum function. The first few rows have been done for you.

x	$f(x) = 3x - 6$	$g(x) = x + 2$	$(f + g)(x)$
-2	-12	0	-12
-1	-9	1	-8
0	***-6***	***2***	***-4***
1	***-3***	***3***	***0***
2	***0***	***4***	***4***
3	***3***	***5***	***8***

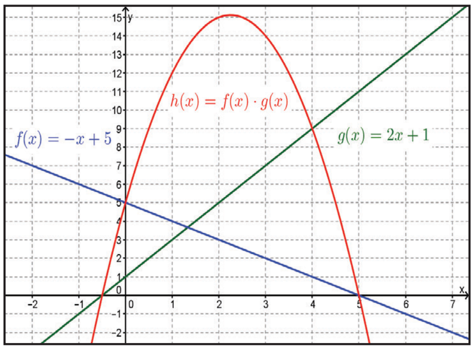
3. Write a polynomial function for $(f + g)(x)$ using the patterns in the y-values in the table. What type of function is this?
The function (f + g)(x) = 4x – 4; which is a linear function.

4. Compare the slope of $(f + g)(x)$ with the slopes of $f(x)$ and $g(x)$.
The slope of (f + g)(x) is the sum of the slopes of f(x) and of g(x): 3 + 1 = 4.

5. Find the y-intercepts for each of the functions. What do you notice about the y-intercepts?
See margin.

6. Find the x-intercepts for each of the functions. What do you notice about the x-intercepts?
See margin.

7. The graph of $f(x) = -x + 5$, $g(x) = 2x + 1$, and $h(x) = f(x) \cdot g(x)$ is shown below.



A. Both $f(x)$ and $g(x)$ are linear functions. What type of function is $h(x) = f(x) \cdot g(x)$?
The function h(x) is a quadratic function.

B. What are the x-intercepts of $f(x)$ and $g(x)$?
See margin.

C. What are the x-intercepts of $h(x)$? How do these compare with the x-intercepts of $f(x)$ and $g(x)$?
See margin.

5. *The y-intercept of (f + g)(x) = (0, -4). The y-intercept of f(x) is (0, -6) and of g(x) is (0, 2). The y-coordinate of the y-intercept of (f + g)(x) is the sum of the y-coordinates of the y-intercepts for the addend functions.*

6. *For (f + g)(x), the x-intercept is (1, 0). For x = 1, the y-coordinates for the addend functions are opposites, -3 and 3, and so their sum of 0 is the y-coordinate for the x-intercept for (f + g)(x).*

For f(x), the x-intercept is (2, 0). The y-coordinates of (f + g)(x) and g(x) are both 4 for x = 2. So the graphs of (f + g)(x) and g(x) would intersect at this point.

For g(x), the x-intercept is (-2, 0). The y-coordinates of (f + g)(x) and f(x) are both -12 for x = -2. So the graphs of (f + g)(x) and f(x) would intersect at this point.

7B. *The x-intercept of f(x) is (5, 0), and the x-intercept of g(x) seems to be at (-0.5, 0).*

7C. *The x-intercepts of h(x) are (5, 0) and (-0.5, 0). They are the same as the x-intercepts of f(x) and g(x).*

Use the following information to answer questions 8 – 11.

Two linear functions, $f(x) = x + 3$ and $g(x) = -2x + 4$, are multiplied to create a new function $(f \cdot g)(x)$.

8. Complete a table of values for $f(x)$, $g(x)$, and the resulting product function. The first few rows have been done for you.

x	$f(x) = x + 3$	$g(x) = -2x + 4$	$(f \cdot g)(x)$
-4	-1	12	-12
-3	0	10	0
-2	***1***	***8***	***8***
-1	***2***	***6***	***12***
0	***3***	***4***	***12***
1	***4***	***2***	***8***
2	***5***	***0***	***0***

9. Write a polynomial function for $(f \cdot g)(x)$ using the patterns in the y-values in the table. What type of function is this?
The function $(f \cdot g)(x) = -2x^2 - 2x + 12$ and is a quadratic function.

10. Find the y-intercepts for each of the functions.
See margin.

11. Find the x-intercepts for each of the functions. What do you notice about the x-intercepts?
See margin.

Use the following information to answer questions 12 – 15.

Two linear functions, $p(x) = -3x + 6$ and $q(x) = 3x - 6$, are multiplied to create a new function $(p \cdot q)(x) = p(x) \cdot q(x)$.

12. Complete the table of values.

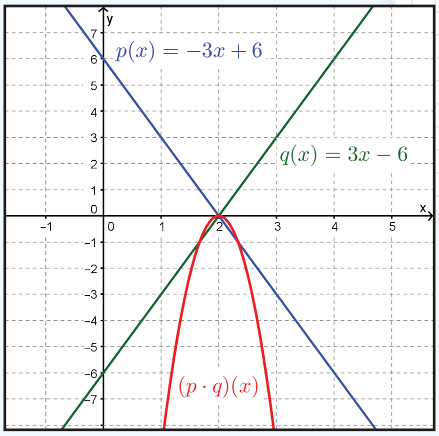
x	$p(x)$	$q(x)$	$p(x) \cdot q(x)$
0	***6***	***-6***	***-36***
1	***3***	***-3***	***-9***
2	***0***	***0***	***0***
3	***-3***	***3***	***-9***
4	***-6***	***6***	***-36***
5	***-9***	***9***	***-81***

13. Use some of the points in the table to graph all three functions in the same coordinate plane.
See margin.

10. *The y-intercept of $(f \cdot g)(x) = (0, 12)$. The y-intercept of $f(x)$ is $(0, 3)$ and of $g(x)$ is $(0, 4)$. The y-coordinate of the y-intercept of $(f \cdot g)(x)$ is the product of the y-coordinates of the y-intercepts for $f(x)$ and $g(x)$.*

11. *The x-intercept of $f(x)$ is $(-3, 0)$ and the x-intercept of $g(x)$ is $(2, 0)$. The x-intercepts of $(f \cdot g)(x)$ are $(-3, 0)$ and $(2,0)$, which are the same as the x-intercepts of the factor functions.*

13.



14. Use the polynomials for the factor functions to write a polynomial for the product function.
$(p \cdot q)(x) = -9x^2 + 36x - 36$

15. Compare the x- and y-intercepts of the factor functions with the intercepts of the product function.
See margin.

Use the following information to answer questions 16 – 19.

Two linear functions, $f(x) = 2x - 1$ and $g(x) = x + 2$, are added and multiplied to create two new functions: $(f + g)(x)$ and $(f \cdot g)(x)$.

16. Complete the table of values for $f(x)$, $g(x)$, and their sum and product functions. The first few rows have been done for you.

x	$f(x)$	$g(x)$	$f(x) + g(x)$	$f(x) \cdot g(x)$
-2	-5	0	-5	0
-1.5	-4	0.5	-3.5	-2
-1	***-3***	***1***	***-2***	***-3***
-0.5	***-2***	***1.5***	***-0.5***	***-3***
0	***-1***	***2***	***1***	***-2***
0.5	***0***	***2.5***	***2.5***	***0***
1	***1***	***3***	***4***	***3***

17. Use the polynomials for $f(x)$ and $g(x)$ to write polynomial functions for the sum and product functions.
$f(x) + g(x) = 3x + 1$ and $f(x) \cdot g(x) = 2x^2 + 3x - 2$

18. What type of function is the sum function $f(x) + g(x)$? What type of function is the product function $f(x) \cdot g(x)$?
The sum function $f(x) + g(x)$ is linear. The product function $f(x) \cdot g(x)$ is quadratic.

19. Compare the x-intercepts, y-intercepts and slopes of the functions, where appropriate.
See margin.

15. *The graphs of all three functions intersect at (2, 0), which means that $(p \cdot q)(x)$ has two identical x-intercepts.*

The y-intercept of $p(x)$ is (0, 6) and of $q(x)$ is (0, -6), which is seen as b in their equations. The y-coordinate of the y-intercept of $(p \cdot q)(x)$ is the product of the y-coordinates of the y-intercepts of the factors, so it is -36.

19. *See margin below.*

19. *This table shows the functions' intercepts and slopes.*

POLYNOMIAL FUNCTION	x-INTERCEPT	y-INTERCEPT	SLOPE
$f(x)$	***(0.5, 0)***	***(0, -1)***	***2***
$g(x)$	***(-2, 0)***	***(0, 2)***	***1***
$(f + g)(x)$	***Approximately (0, -0.33) or $(-\frac{1}{3}, 0)$***	***(0, 1)***	***3***
$(f \cdot g)(x)$	***(0.5, 0) and (-2, 0)***	***(0, -2)***	***N/A***

For questions 20 – 22, determine the polynomial functions for $(f + g)(x)$ and $(f \cdot g)(x)$. Use the function equations, tables, or graphs to compare $f(x)$ and $g(x)$ with both their sum and product functions. Identify the x-intercepts, y-intercepts, and slopes (where appropriate) of all functions.

20. $f(x) = 2x - 1$ and $g(x)= -3x$

POLYNOMIAL FUNCTION	x-INTERCEPT	y-INTERCEPT	SLOPE
$f(x) = 2x - 1$	***(0.5, 0)***	***(0, -1)***	***2***
$g(x) = -3x$	***(0, 0)***	***(0, 0)***	***-3***
$(f + g)(x) = -x - 1$	***(-1, 0)***	***(0, -1)***	***-1***
$(f \cdot g)(x) = -6x^2 + 3x$	***(0.5, 0) and (0, 0)***	***(0, 0)***	***N/A***

21. $f(x) = x - 5$ and $g(x)= 3x + 9$

POLYNOMIAL FUNCTION	x-INTERCEPT	y-INTERCEPT	SLOPE
$f(x) = x - 5$	***(5, 0)***	***(0, -5)***	***1***
$g(x) = 3x + 9$	***(-3, 0)***	***(0, 9)***	***3***
$(f + g)(x) = 4x + 4$	***(-1, 0)***	***(0, 4)***	***4***
$(f \cdot g)(x) = 3x^2 - 6x - 45$	***(5, 0) and (-3, 0)***	***(0, -45)***	***N/A***

22. $f(x) = 2x + 3$ and $g(x)= 2x - 4$

POLYNOMIAL FUNCTION	x-INTERCEPT	y-INTERCEPT	SLOPE
$f(x) = 2x + 3$	***(-1.5, 0)***	***(0, 3)***	***2***
$g(x) = 2x - 4$	***(2, 0)***	***(0, -4)***	***2***
$(f + g)(x) = 4x - 1$	***(0.25, 0)***	***(0,-1)***	***4***
$(f \cdot g)(x) = 4x^2 - 2x - 12$	***(-1.5, 0) AND (2, 0)***	***(0, -12)***	***N/A***