Dividing Polynomial Functions with Algebraic Methods

5.10

FOCUSING QUESTION How can you use algebraic methods to divide polynomial functions?

LEARNING OUTCOMES

- I can use algebraic methods to divide two polynomial functions.
- I can apply mathematics to solve problems arising in society.

ENGAGE

According to Ohm's Law, the voltage, *V*, in an electrical circuit is equal to the product of the current, *I*, and the resistance, *R*, in the circuit.

$$V = IR$$

If the voltage in a particular circuit is equal to $V(x) = 2x^2 + 7x - 4$ volts and the current is equal to $I(x) = x + 4$ amperes, what function is $R(x)$, the resistance in the circuit in ohms?

$R(x) = 2x - 1$

EXPLORE

Each year, a different number of immigrants move to the United States from other countries. The table contains data describing the number of immigrants arriving in the United States according to the U.S. census from 1930 to 2000. Demographers, or people who study population statistics, use mathematical models such as polynomial functions to analyze data and make predictions.

YEAR	NUMBER OF IMMIGRANTS	U.S. POPULATION	PERCENT OF POPULATION
1930	14,204,100	123,202,624	11.6
1940	11,594,900	132,164,569	8.8
1950	10,347,400	151,325,798	6.9
1960	9,738,100	179,323,175	5.4
1970	9,619,300	203,211,926	4.7
1980	14,079,900	226,545,805	6.2
1990	19,767,300	248,765,170	7.9
2000	31,107,900	281,421,906	11.1

TEKS

AR.4C Determine the quotient of a polynomial function of degree three and of degree four when divided by a polynomial function of degree one and of degree two when represented tabularly and symbolically.

MATHEMATICAL PROCESS SPOTLIGHT

AR.1A Apply mathematics to problems arising in everyday life, society, and the workplace.

ELPS

3H Narrate, describe, and explain with increasing specificity and detail as more English is acquired.

VOCABULARY

quotient, dividend, divisor, synthetic division

MATERIALS

- graph paper

Let x represent the number of decades (10-year periods) since 1930. Use a problem-solving model to generate and compare function models for $m(x)$, the number of immigrants (in millions) moving to the U.S. x decades after 1930, $p(x)$, the U.S. population (in millions) x decades after 1930, and $q(x)$, the percent of the U.S. population made up of immigrants x decades after 1930.

1. Make a scatterplot of each of the following
- Number of immigrants (in millions) versus number of decades after 1930
- U.S. population (in millions) versus number of decades after 1930
- Percent of population versus number of decades after 1930

See margin.

2. What type of function do you think could model each scatterplot? Explain your reasoning.
See margin.

3. A percent is a ratio of the size of a part to the size of the whole. How would you calculate the percent of the U.S. population made up of immigrants in any year from the U.S. population and the number of immigrants?
See margin.

The function $m(x) = 0.12x^3 - 0.348x^2 - 1.116x + 12.96$ closely models the number of millions of immigrants, $m(x)$, in any decade since 1930, x. The function $p(x) = 24x + 108$ closely models the U.S. population in millions, $p(x)$, in any decade since 1930, x.

4. Write the function $q(x)$, the percent of the U.S. population made up of immigrants in any decade since 1930, x, as a ratio of $m(x)$ and $p(x)$ using the definition of a percent.

$$q(x) = \frac{0.12x^3 - 0.348x^2 - 1.116x + 12.96}{24x + 108} \times 100$$

5. Use your ratio to identify the dividend and divisor for the quotient, $q(x)$.
The dividend is $m(x) = 0.12x^3 - 0.348x^2 - 1.116x + 12.96$ and the divisor is $p(x) = 24x + 108$.

Division can be represented with a division bracket.

$$\begin{array}{r} \text{quotient} \\ \text{divisor}\overline{)\text{dividend}} \end{array}$$

Division with polynomial functions follows the same pattern. The function $q(x)$ can be written as

$$24x + 108\overline{)0.12x^3 - 0.348x^2 - 1.116x + 12.96}$$

Whole numbers and decimal numbers are divided using a long-division standard algorithm. In this algorithm, place value is used to determine how many groupings of the divisor there are in each digit of the dividend. For example, 6 does not divide into 3 hundreds, but 6 does divide into 32 tens. A 5 is recorded in the tens place of the quotient, directly above the tens digit of the dividend. 6 times 5 tens is 30 tens, so 30 tens are subtracted from the 32 tens in the dividend, leaving 2 tens remaining. The 4 ones are brought down and 6 divides into 24 ones 4 times. A 4 is recorded in the ones place of the quotient, directly above the ones digit of the dividend. 24 ones are subtracted from 24 ones in the dividend, leaving 0 remaining.

$$\begin{array}{r} 54 \\ 6\overline{)\,324} \\ \underline{-30} \\ 24 \\ \underline{-24} \\ 0 \end{array}$$

1. *See margin below.*

2. *Answers may vary. Possible response:*

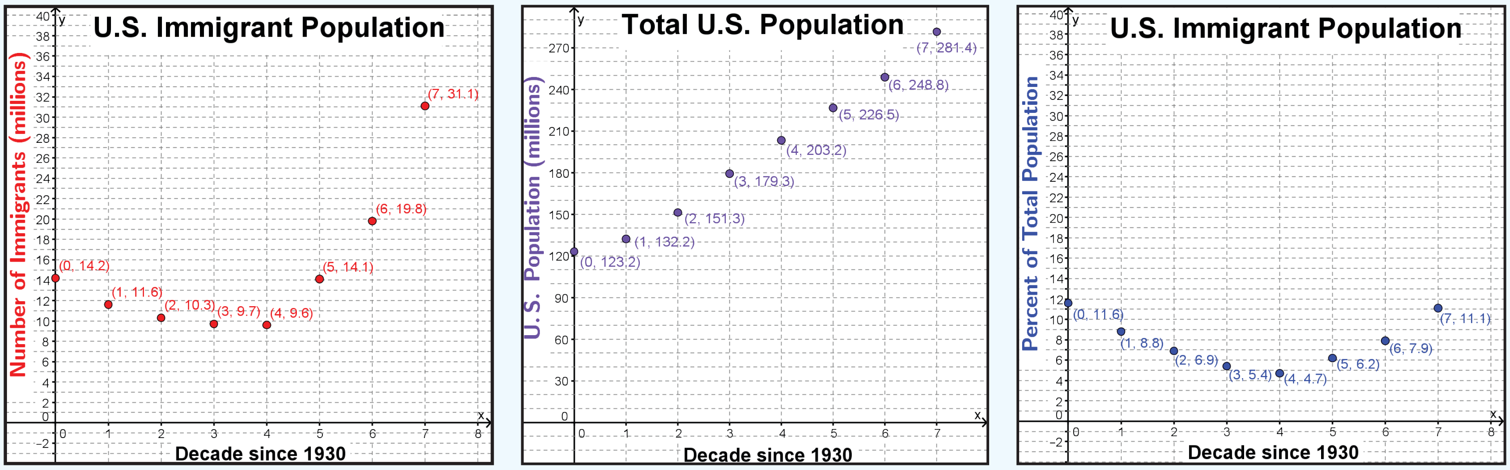
The scatterplot of number of immigrants versus number of decades appears to be a cubic function because it decreases slowly then begins to increase quickly.

The scatterplot of U.S. population versus number of decades appears to be linear because there is an approximately constant slope between consecutive points.

The scatterplot of percent of population versus number of decades appears to be quadratic because it decreases, levels out, and then increases at a similar rate as the original decrease.

3. *The number of immigrants is a part of the whole U.S. population, so the percent of the U.S. population made up of immigrants is* $\frac{\text{Number of Immigrants}}{\text{U.S.Population}} \times 100$.

1.



6. Compare the first term in the divisor with the first term in the dividend. What can be multiplied by $24x$ in order to generate a product of $0.12x^3$? Multiply that factor by both terms of the divisor. Record the resulting x^3 term beneath the x^3 term of the dividend. Record the resulting x^2 term beneath the x^2 term of the dividend. Subtract these new terms from the terms in the dividend.

$$\begin{array}{r} 0.005x^2 \qquad\qquad\qquad\qquad\qquad\quad \\ 24x + 108 \overline{\smash{)}\, 0.12x^3 - 0.348x^2 - 1.116x + 12.96} \\ \underline{-(0.12x^3 + \mathbf{0.54x^2})} \qquad\qquad\qquad\quad \\ \mathbf{0x^3 - 0.888x^2} \qquad\qquad\qquad\quad \end{array}$$

Think $\frac{0.12x^3}{24x} = 0.005x^2$, so record $0.005x^2$ directly above $0.12x^3$ in the dividend.

7. Compare the first term in the divisor with the first non-zero term in the remaining portion of the dividend. What can be multiplied by $24x$ in order to generate a product of $-0.888x^2$? Multiply that factor by both terms of the divisor. Record the resulting x^2 term beneath the x^2 term of the dividend. Record the resulting x term beneath the x term of the dividend. Subtract these new terms from the terms in the dividend.

$$\begin{array}{r} 0.005x^2 - 0.037x \qquad\qquad\qquad\quad \\ 24x + 108 \overline{\smash{)}\, 0.12x^3 - 0.348x^2 - 1.116x + 12.96} \\ \underline{-(0.12x^3 + 0.54x^2)} \qquad\qquad\qquad\quad \\ -0.888x^2 \qquad\qquad\qquad\quad \\ \underline{-(-0.888x^2 \mathbf{- 3.996x})} \qquad\qquad \\ \mathbf{0x^2 + 2.88x} \qquad\qquad \end{array}$$

Think $\frac{-0.888x^2}{24x} = -0.037x$, so record $-0.037x$ directly above $-0.348x^2$ in the dividend.

8. Compare the first term in the divisor with the first non-zero term in the remaining portion of the dividend. What can be multiplied by $24x$ in order to generate a product of $2.88x$? Multiply that factor by both terms of the divisor. Record the resulting x term beneath the x term of the dividend. Record the resulting constant term beneath the constant term of the dividend. Subtract these new terms from the terms in the dividend.

$$\begin{array}{r} 0.005x^2 - 0.037x + 0.12 \qquad\qquad\quad \\ 24x + 108 \overline{\smash{)}\, 0.12x^3 - 0.348x^2 - 1.116x + 12.96} \\ \underline{-(0.12x^3 + 0.54x^2)} \qquad\qquad\qquad\quad \\ -0.888x^2 \qquad\qquad\qquad\quad \\ \underline{-(-0.888x^2 - 3.996x)} \qquad\qquad \\ 2.88x \qquad\qquad \\ \underline{-(2.88 \mathbf{+ 12.96})} \\ 0 \qquad \end{array}$$

Think $\frac{2.88x}{24x} = 0.12$, so record $+0.12$ directly above $-1.116x$ in the dividend.

9. The function $q(x) = 0.005x^2 - 0.037x + 0.12$ represents the ratio of the number of immigrants (in millions) to the U.S. total population (in millions) as a decimal number. Multiply $q(x)$ by 100 in order to generate $q(x)$ as the percent of the U.S. population that is made up of immigrants.
$q(x) = 0.5x^2 - 3.7x + 12$ percent

SUPPORTING ENGLISH LANGUAGE LEARNERS

When students are explaining their reasoning (**ELPS 3H**), monitor to make sure that they are explaining with increasing specificity and detail as more English is acquired. For all students, listen for their reasoning about why the scatterplot resembles the function type that it does.

- If a student suggests that the number of immigrants versus number of decades appears to be exponential, ask them to explain the data values for decades 0 through 4. Exponential functions either only increase or only decrease.
- If a student suggests that the number of immigrants versus number of decades appears to be quadratic, ask them to identify the vertex. How do the rates of decrease and increase on either side of the vertex compare? Quadratic functions decrease and then increase at the same rates on either side of the vertex.

10. *For 2010, $x = 8$*

$m(8) = 43.2$ million

$p(8) = 300$ million

$q(8) = 14.4\%$

11. *In the predictions, the number of immigrants and the percent of the population were overestimated and the total U.S. population was underestimated.*

INSTRUCTIONAL HINT

As students move through the procedure for dividing polynomial functions, help them make connections to the division algorithm for whole numbers and decimals.

REFLECT ANSWER:

Both procedures are based on place value. Division with whole numbers uses base-10 place value (ones, tens, hundreds, thousands, etc.) which are powers of 10. Division with polynomial functions uses base-x place value (constant (x^0), x, x^2, x^3, etc.) which are powers of x.

10. Use the functions $m(x)$, $p(x)$, and $q(x)$ to predict the number of immigrants living in the U.S., the total U.S. population, and the percent of the U.S. population made up of immigrants in 2010.
See margin.

11. According to the U.S. Census Bureau, in 2010, the total U.S. population was 308,745,538. Of these, 39,955,900 million were immigrants, which is 12.9% of the U.S. population. How do the actual figures compare to those predicted by your functions?
See margin.

REFLECT

- **How is the long division algorithm for dividing polynomial functions similar to the long division algorithm for dividing whole numbers?**
  See margin.

$$\begin{array}{r} 54 \\ 6\overline{)\,324} \\ \underline{-30} \\ 24 \\ \underline{-24} \\ 0 \end{array}$$

- **When dividing whole numbers, a remainder can be expressed as a fraction.**

$$329 \div 6 = 54 \text{ R } 5 = 54\tfrac{5}{6}$$

If you are dividing polynomial functions and there is a remainder, how do you think you would express the remainder as a fraction?
The remainder becomes the numerator and the divisor becomes the denominator.

EXPLAIN

There are several ways to divide polynomial functions using algebraic methods.

Watch Explain and You Try It Videos

or click here

DIVIDING POLYNOMIALS USING LONG DIVISION

The long division algorithm, or standard algorithm, used to divide whole numbers and decimal numbers can also be used to divide polynomials and polynomial functions. Let's compare how the place value system is used to organize both algorithms.

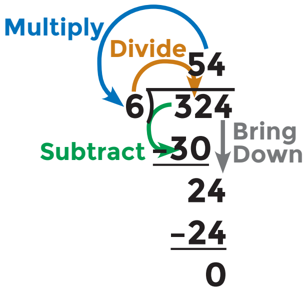
			x^2	x	c		
x^2	x	c	x^4	x^3	x^2	x	c
			x^4	x^3	x^2		
				x^3	x^2	x	
				x^3	x^2	x	
					x^2	x	c
					x^2	x	c
						x	c

			10^2	10	1		
10^2	10	1	10^4	10^3	10^2	10	1
			10^4	10^3	10^2		
				10^3	10^2	10	
				10^3	10^2	10	
					10^2	10	1
					10^2	10	1
						10	1

In both algorithms, terms are lined up with the greatest power to the left and listed in descending order. For polynomials and polynomial functions, the base is *x*. For whole numbers and decimals, the base is 10. Whole numbers or decimals written in expanded notation are similar to polynomials.

$$632 = 6(10)^2 + 3(10)^1 + 2(10)^0$$
$$6x^2 + 3x^1 + 2x^0$$
$$6x^2 + 3x + 2$$

Using the algorithm for whole numbers consists of a cycle of four steps.



1. **Divide** the divisor into the first few digits of the dividend. Record the partial quotient in the appropriate place value of the quotient.
2. **Multiply** the divisor by the digit recorded in the quotient. Record the product beneath the dividend in the appropriate place value columns.
3. **Subtract** the product from the first part of the dividend. Record the difference beneath the product.
4. **Bring Down** the next digit from the dividend. Be sure to align place value.

Repeat the process.

The algorithm for dividing polynomial functions uses the same cycle of four steps.

Divide 0.12*x*³ by 24*x* to get 0.005*x*².
Multiply 0.005*x*² by 24*x* and 108.
Subtract 0.12*x*³ and 0.54*x*².
Bring down -1.116*x*.

Divide -0.888*x*² by 24*x* to get -0.037*x*.
Multiply -0.037*x* by 24*x* and 108.
Subtract -0.888*x*² and -3.996*x*.
Bring down 12.96.

Divide 2.88*x* by 24*x* to get 0.12.
Multiply 0.12 by 24*x* and 108.
Subtract 2.88*x* and 12.96.
The difference is 0 so the process ends.

$$\begin{array}{r} 0.005x^2 - 0.037x + 0.12 \\ 24x + 108 \overline{\big) \; 0.12x^3 - 0.348x^2 - 1.116x + 12.96} \\ \underline{-(0.12x^3 + 0.54x^2)} \qquad\qquad\qquad\qquad \\ -0.888x^2 \qquad\qquad\qquad\quad \\ \underline{-(-0.888x^2 - 3.996x)} \qquad\quad \\ 2.88x \qquad\qquad \\ \underline{-(2.88 + 12.96)} \\ 0 \end{array}$$

INSTRUCTIONAL HINT

For students struggling with long division, set up a “speed division” game. Have students face off with a similarly matched partner to see who can solve a long division problem, like 324 divided by 6, the quickest with total accuracy. This will allow for students to practice multiple problems to recall their long division skills in a more engaging fashion.

As with whole numbers, all places with polynomial functions must be accounted for.

$$(6 \times 1000) + (4 \times 10) + (7 \times 1)$$
$$= 6(10)^3 + 0(10)^2 + 4(10)^1 + 7(10)^0$$
$$= 6{,}047$$

$$6x^3 + 4x + 7$$
$$=6(x)^3 + 0(x)^2 + 4(x)^1 + 7x$$
$$= 6x^3 + 0x^2 + 4x + 7$$

When dividing whole numbers, a 0 is used as a placeholder to ensure that all place values that are powers of 10 are accounted for. When dividing polynomials, a missing term is represented with a 0 coefficient to serve as a placeholder to ensure that all place values that are powers of x are accounted for.

ANOTHER STRATEGY: SYNTHETIC DIVISION

Another algebraic method that you can use to divide polynomial functions is called **synthetic division**. If you have a polynomial function and you know one linear factor, then you can use synthetic division to divide the polynomial function by the linear function representing the linear factor.

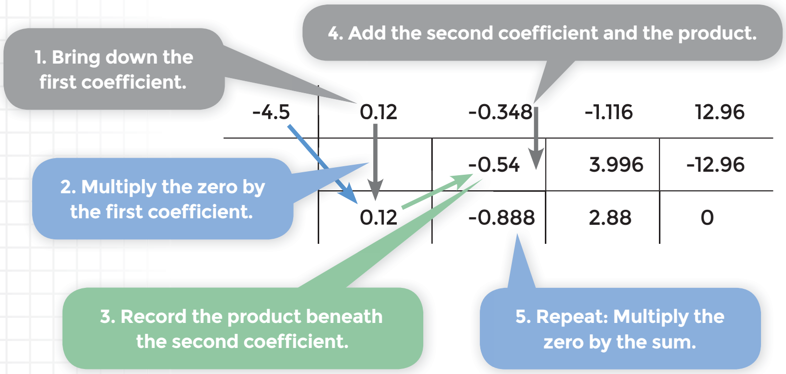
Synthetic division assumes a zero from a linear factor of a polynomial function. The structure of synthetic division uses coefficients of a polynomial function, $z(x) = ax^4 + bx^3 + cx^2 + dx + f$ with a zero of from the linear factor $ax - b$.

$\frac{b}{a}$	a	b	c	d	f

For the polynomial functions $m(x) = 0.12x^3 - 0.348x^2 - 1.116x + 12.96$ and $p(x) = 24x + 108$, the quotient $m(x) \div p(x)$ is a cubic function divided by a linear function. You can use the linear function to identify a possible 0 of the dividend, $m(x)$.

$$24x + 108 = 0$$
$$24x = -108$$
$$x = \frac{-108}{24} = -4.5$$

Once the zero has been identified, substitute it along with the coefficients of $m(x)$, in descending order of exponents, into the synthetic division table. If a term is missing from the polynomial, include it with a 0 coefficient.



-4.5	0.12	-0.348	-1.116	12.96
		-0.54	3.996	-12.96
	0.12	-0.888	2.88	0

1. Bring down the first coefficient.

2. Multiply the zero by the first coefficient.

3. Record the product beneath the second coefficient.

4. Add the second coefficient and the product.

5. Repeat: Multiply the zero by the sum.

Synthetic division is particularly useful for coding since it isolates the coefficients of polynomial functions and employs a cycle of set procedures in order to divide two polynomial functions.

Since $m(x)$ was a cubic function that was divided by $p(x)$, which was a linear function, the degree of $q(x)$ is 3 – 1 = 2, so $q(x)$ is a quadratic function. The coefficients in the bottom row of the synthetic division table can be used to write $q(x)$ as a quadratic function in polynomial form.

$$q(x) = 0.12x^2 - 0.888x + 2.88$$

Lastly, since the linear factor, $p(x) = 24x + 108 = 24(x + 4.5)$, had a coefficient of x that was not equal to 1, divide each term by the coefficient, 24.

$$q(x) = \frac{0.12}{24}x^2 - \frac{0.888}{24}x + \frac{2.88}{24}$$
$$q(x) = 0.005x^2 - 0.037x + 0.12$$

Notice that $q(x)$ obtained through synthetic division is equivalent to $q(x)$ obtained through the long division algorithm.

DIVIDING POLYNOMIAL FUNCTIONS USING ALGEBRAIC METHODS

Polynomial functions have a place value system that is based on powers of a variable. Compare this place value system to base-10 numbers that are used for whole numbers and decimals. Base-10 numbers have a place value system based on powers of 10. So, algorithms for whole numbers and decimals can be extended to polynomial functions.

Long Division Algorithm

- Write the dividend and divisor with terms in descending exponential order. Use a 0-coefficient to create placeholders for missing terms.
  - ✓ Divide the leading term of the divisor into the leading term of the dividend. Record this number above the leading term of the dividend in the quotient.
  - ✓ Multiply all terms of the divisor by the number placed in the quotient. Record these terms beneath their appropriate place value (e.g., all x^3 terms should line up vertically).
  - ✓ Subtract all new terms. Record the differences in the correct place value column.
  - ✓ Bring down the next term from the dividend.
  - ✓ Repeat the cycle until all terms in the dividend have been exhausted.

INSTRUCTIONAL HINT

Encourage students to add the steps for long division and synthetic division to their notebook to assist them as they practice these types of problems.

DIVIDING POLYNOMIAL FUNCTIONS USING ALGEBRAIC METHODS

Synthetic Division

- Use the coefficients of the dividend and linear function divisor to create a synthetic division table. Make sure the terms are in descending exponential order. Use a 0-coefficient to create placeholders for missing terms.
  - ✓ Use the divisor to identify a zero from the linear function. Record this zero in the top left corner of the synthetic division table.
  - ✓ Bring down the lead coefficient to the bottom row of the synthetic division table. Record the coefficients of the polynomial in the synthetic division table being sure to use a 0-coefficient for any missing terms.
  - ✓ Multiply the lead coefficient by the zero. Record this product beneath the next coefficient.
  - ✓ Add the second coefficient column. Record this sum in the bottom row of the synthetic division table.
  - ✓ Multiply the sum by the zero. Record this product beneath the next coefficient.
  - ✓ Repeat the cycle until all terms in the dividend have been exhausted.

EXAMPLE 1

If the volume of a wedge of cheese is represented by $V(x) = x^3 + 13x^2 + 44x + 12$ and its height is $h(x) = x + 6$, find the area of its base using the formula $V(x) = A(x) \cdot h(x)$.

Image source: openclipart.org

STEP 1 **Determine what operation to use.**

Since $V(x) = A(x) \cdot h(x)$, you can find $A(x)$ by dividing both sides of the formula by $h(x)$. So $\frac{V(x)}{h(x)}$. So then $A(x) = \frac{V(x)}{h(x)} = \frac{x^3 + 13x^2 + 44x + 12}{x + 6}$.

STEP 2 **Perform the division process using the long division algorithm.**

Divide x into x^3, multiply x^2 by x + 6, subtract the product, and bring down $44x$.

$$\begin{array}{r} x^2 \\ x+6\,\overline{\smash{)}\,x^3+13x^2+44x+12} \\ \underline{(x^3+6x^2)} \\ 7x^2+44x \end{array}$$

Divide x into $7x^2$, multiply $7x$ by $x + 6$, subtract the product, and bring down 12.

$$\begin{array}{r} x^2 \;+\; 7x \\ x+6\,\overline{\smash{)}\,x^3+13x^2+44x+12} \\ \underline{(x^3+6x^2)} \\ 7x^2+44x \\ \underline{(7x^2+42x)} \\ 2x+12 \end{array}$$

Complete the division by dividing x into $2x$, multiplying and subtracting as before.

$$\begin{array}{r} x^2 \;+\; 7x \;+\; 2 \\ x+6\,\overline{\smash{)}\,x^3+13x^2+44x+12} \\ \underline{(x^3+6x^2)} \\ 7x^2+44x \\ \underline{(7x^2+42x)} \\ 2x+12 \\ \underline{(2x+12)} \\ 0 \end{array}$$

STEP 3 **Interpret the quotient you've found.**

Since there is no remainder, the polynomial result, $x^2 + 7x + 2$, represents the area of the base, $A(x)$.

YOU TRY IT! #1

Divide $5x^3 + 39x^2 + 31x + 21$ by $x + 7$, and evaluate your answer for reasonableness.
See margin.

ADDITIONAL EXAMPLES

For the functions $f(x)$ and $g(x)$, divide $f(x)$ by $g(x)$, and evaluate your answer for reasonableness.

1. $f(x) = 8x^3 + 14x^2 - 29x - 35$ and $g(x) = 2x + 5$

The quotient is $4x^2 - 3x - 7$.

2. $f(x) = -15x^3 + 53x^2 - 78x + 84$ and $g(x) = 3x - 7$

The quotient is $-5x^2 + 6x - 12$.

3. $f(x) = -10x^3 + 22x^2 - 29x + 5$ and $g(x) = -5x + 1$

The quotient is $2x^2 - 4x + 5$.

YOU TRY IT! #1 ANSWER:

Because there was no remainder, the quotient, $5x^2 + 4x + 3$, *is a polynomial. It is reasonable that a cubic polynomial divided by a linear would result in a quadratic. For a check on the division, you can multiply the quotient,* $5x^2 + 4x + 3$, *by the divisor,* $x + 7$, *which results in the dividend,* $5x^3 + 39x^2 + 31x + 21$.

EXAMPLE 2

Paxton is buying a water tank to carry in his truck. The tanks are cylindrical and the volume is represented by the cubic function $V(x) = 200x^3 - 35x^2 - 63x + 18$. Paxton needs to find a tank that will fit across the bed of his truck, which measures 4 ft. 4 in. He reasons that the height of the tank can be found by dividing its volume by the area of the base of the cylinder. The area of the base is represented by the quadratic function $A(x) = 40x^2 - 31x + 6$. Find the height of the tank in feet, $h(x)$, using standard long division. Using the quotient, determine the value(s) of x that generate the greatest height of a tank that will fit in Paxton's truck bed.

STEP 1 **Set up the division algorithm, accounting for all the terms in both the dividend and divisor. Divide $40x^2$ into $200x^3$ and record the quotient above.**

$$\begin{array}{rl} & 5x \\ 40x^2 - 31x + 6 & \overline{)\,200x^3 - 35x^2 - 63x + 18} \end{array}$$

Step 2 **Multiply $5x$ by $40x^2 - 31x + 6$ and record the product below the same powers of x in the dividend.**

$$\begin{array}{rl} & 5x \\ 40x^2 - 31x + 6 & \overline{)\,200x^3 - 35x^2 - 63x + 18} \\ & 200x^3 - 155x^2 + 30x \end{array}$$

STEP 3 **Subtract the products and bring down the 18.**

$$\begin{array}{rl} & 5x \\ 40x^2 - 31x + 6 & \overline{)\,200x^3 - 35x^2 - 63x + 18} \\ & \underline{-(200x^3 - 155x^2 + 30x)} \\ & 120x^2 - 93x + 18 \end{array}$$

STEP 4 **Divide $40x^2$ into $120x^2$. Record the 3 next to the $5x$.**

$$\begin{array}{rl} & 5x + 3 \\ 40x^2 - 31x + 6 & \overline{)\,200x^3 - 35x^2 - 63x + 18} \\ & \underline{-(200x^3 - 155x^2 + 30x)} \\ & 120x^2 - 93x + 18 \end{array}$$

STEP 5 Multiply the whole divisor by 3, record it under the same powers of x, and subtract.

$$
\begin{array}{r}
5x \;+\; 3 \\
40x^2 - 31x + 6\overline{\smash{)}\,200x^3 - 35x^2 - 63x + 18} \\
\underline{-\,(200x^3 - 155x^2 + 30x)} \\
120x^2 - 93x + 18 \\
\underline{-\,(120x^2 - 93x + 18)} \\
0
\end{array}
$$

STEP 6 Use the polynomial function, $h(x) = 5x + 3$, representing the height of a tank, to determine which value of x will generate a value of 4 feet 4 inches, or $4\frac{1}{3}$ feet.

$$
\begin{aligned}
4\tfrac{1}{3} &= 5x + 3 \\
4\tfrac{1}{3} - 3 &= 5x + 3 - 3 \\
1\tfrac{1}{3} &= 5x \\
\frac{1\frac{1}{3}}{5} &= \frac{5x}{5} \\
\frac{4}{15} &= x
\end{aligned}
$$

YOU TRY IT! #2

Determine if $0.1x^2 - 6x + 0.36$ is a factor of $-0.1x^3 + 6.05x^2 - 3.36x + 0.18$. Use the standard long division algorithm.

Because $-0.1x^3 + 6.05x^2 - 3.36x + 0.18$ divided by $0.1x^2 - 6x + 0.36$ is $-x + 0.5$ with no remainder, $0.1x^2 - 6x + 0.36$ is a factor of $-0.1x^3 + 6.05x^2 - 3.36x + 0.18$.

EXAMPLE 3

Divide the quartic function $f(x) = 5x^4 + 52x^3 + 145x^2 + 50x$ by the linear function $g(x) = 5x + 2$ using synthetic division and determine if the quotient function $h(x)$ is a polynomial.

STEP 1 Use the divisor to identify a zero from the linear function.

$$
\begin{aligned}
5x + 2 &= 0 \\
5x + 2 - 2 &= 0 - 2 \\
5x &= -2 \\
\frac{5x}{5} &= \frac{-2}{5} \\
x &= -\frac{2}{5} = -0.4
\end{aligned}
$$

ADDITIONAL EXAMPLES

For the functions $f(x)$ and $g(x)$, divide $f(x)$ by $g(x)$, and evaluate your answer for reasonableness.

1. $f(x) = -6x^3 - 11x^2 - 8x - 3$ and $g(x) = 6x^2 + 5x + 3$

The quotient is $-x - 1$.

2. $f(x) = 14x^3 - 45x^2 - 77x - 18$ and $g(x) = 2x^2 - 7x - 9$

The quotient is $7x + 2$.

3. $f(x) = -3x^3 + 25x + 6$ and $g(x) = -3x^2 - 9x - 2$

The quotient is $x - 3$.

ADDITIONAL EXAMPLES

Find the quotient, $q(x)$, of the quartic function, $f(x)$, and the linear function, $g(x)$, with synthetic division. Use the divisor to identify a zero from the linear function, divide the resulting coefficients if necessary, and determine if the quotient is a polynomial function.

1. $f(x) = 6x^4 + 23x^3 + 38x^2 + 30x + 8$ and $g(x) = 3x + 4$

The zero is $-\frac{4}{3}$. *The quotient is* $q(x) = 2x^3 + 5x^2 + 6x + 2$ *after dividing the bottom row of the synthetic division by 3, and it is a cubic polynomial function.*

2. $f(x) = 5x^4 - 22x^3 + 43x^2 - 19x + 2$ and $g(x) = 5x - 2$

The zero is $\frac{2}{5}$. *The quotient is* $q(x) = x3 - 4x^2 + 7x - 1$ *after dividing the bottom row of the synthetic division by 5, and it is a cubic polynomial function.*

3. $f(x) = -6x^4 - 23x^3 + 8x^2 - 29x + 72$ and $g(x) = 2x + 9$

The zero is $-\frac{9}{2}$. *The quotient is* $q(x) = -3x^3 + 2x^2 - 5x + 8$ *after dividing the bottom row of the synthetic division by 2, and it is a cubic polynomial function.*

STEP 2 Record this zero in the top left corner of the synthetic division table. Write the coefficients of the dividend, making sure the terms are in descending exponential order and using a 0-coefficient for the x^0 or constant term.

-0.4	5	52	145	50	0

STEP 3 Bring down the lead coefficient to the bottom row of the synthetic division table. Multiply the lead coefficient by the zero. Record this product beneath the next coefficient.

-0.4	5	52	145	50	0
		-2			
	5				

STEP 4 Add the second coefficient column. Record this sum in the bottom row of the synthetic division table.

-0.4	5	52	145	50	0
		-2			
	5	50			

STEP 5 Multiply the sum by the zero. Record this product beneath the next coefficient. Repeat the cycle.

-0.4	5	52	145	50	0
		-2	-20	-50	0
	5	50	125	0	0

STEP 6 Interpret the synthetic division table results.

The bottom row indicates the coefficients of the quotient, $5x^3 + 50x^2 + 125x + 0$. There is a zero remainder, so the quotient is a cubic polynomial function. But the process of synthetic division divided the quartic function $f(x)$ by the factor $x + 0.4$, not by $g(x)$. Because the linear divisor, $g(x) = 5x + 2 = 5(x + 0.4)$, has a coefficient of 5, we must divide the quotient by 5 at this point to completely divide $f(x)$ by $g(x)$.

$h(x) = (5x^3 + 50x^2 + 125x + 0) \div 5$
$h(x) = x^3 + 10x^2 + 25x.$

YOU TRY IT! #3

Find the quotient, $q(x)$, of the quartic function, $m(x) = 32x^4 - 360x^3 + 1428x^2 - 2254x + 1029$, and the linear function, $n(x) = 4x - 3$, with the synthetic division method. Use the divisor to identify a zero from the linear function, divide the resulting coefficients if necessary, and determine if the quotient is a polynomial function.

See margin.

EXAMPLE 4

The ideal mechanical advantage of a lever is the ratio between the distance of a load from the fulcrum and the distance of an effort from that fulcrum. Given a situation where the distance of the load is the function $f(x) = x^4 + 4$ and the distance of the effort is the function $l(x) = x^2 - 2x + 2$, find the mechanical advantage, $m(x)$, of the lever.

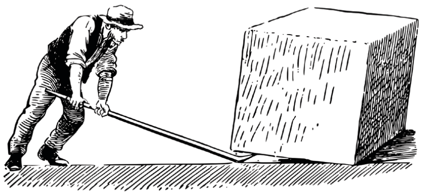


Image source: pixabay.com

STEP 1 Write the ratio of the functions involved.

$$m(x) = \frac{\text{distance of load}}{\text{distance of effort}} = \frac{x^4 + 4}{x^2 - 2x + 2}$$

STEP 2 Divide the numerator by the denominator to find a polynomial function that represents the ratio. Represent any missing terms with zero coefficients.

$$x^2 - 2x + 2\,\overline{)\,x^4 + 0x^3 + 0x^2 + 0x + 4}$$

STEP 3 Begin the "divide – multiply – subtract – bring down" process.

$$\begin{array}{rl}
 & x^2 \\
x^2 - 2x + 2 & \overline{)\,x^4 + 0x^3 + 0x^2 + 0x + 4} \\
 & \underline{-\left(x^4 - 2x^3 + 2x^2\right)} \\
 & \qquad 2x^3 - 2x^2 + 0x
\end{array}$$

STEP 4 Repeat the "divide – multiply – subtract – bring down" process.

$$\begin{array}{rl}
 & x^2 + 2x \\
x^2 - 2x + 2 & \overline{)\,x^4 + 0x^3 + 0x^2 + 0x + 4} \\
 & \underline{-\left(x^4 - 2x^3 + 2x^2\right)} \\
 & \qquad 2x^3 - 2x^2 + 0x \\
 & \qquad \underline{-\left(2x^3 - 4x^2 + 4x\right)} \\
 & \qquad\qquad 2x^2 - 4x + 4
\end{array}$$

YOU TRY IT! #3 ANSWER:

The zero is $\frac{3}{4}$. The quotient is $q(x) = 8x^3 - 84x^2 + 294x - 343$ after dividing the bottom row of the synthetic division table by 4, and it is a cubic polynomial function.

ADDITIONAL EXAMPLES

Using the standard long division algorithm, divide the quartic function, $f(x)$, by the quadratic function, $g(x)$. If possible, express the quotient, $h(x)$, as a polynomial function.

1. $f(x) = x^4 + 3x^3 + 7x^2 + 15x + 10$ and $g(x) = x^2 + 5$

The quotient is $h(x) = x^2 + 3x + 2$ and is a cubic polynomial function.

2. $f(x) = 4x^4 - 25x^2 - 30x - 9$ and $g(x) = 2x^2 + 5x + 3$

The quotient is $h(x) = 2x^2 - 5x - 3$ and is a cubic polynomial function.

3. $f(x) = 9x^4 - 49$ and $g(x) = 3x^2 + 7$

The quotient is $h(x) = 3x^2 - 7$ and is a cubic polynomial function.

STEP 5 Continue the “divide – multiply – subtract – bring down” process until there are no more terms in the dividend to bring down.

$$
\begin{array}{rl}
 & \; x^2 + 2x \\
\cline{2-2}
x^2 - 2x + 2 & \big)\; x^4 + 0x^3 + 0x^2 + 0x + 4 \\
 & -\,(x^4 - 2x^3 + 2x^2) \\
\cline{2-2}
 & \qquad 2x^3 - 2x^2 + 0x \\
 & \qquad -\,(2x^3 - 4x^2 + 4x) \\
\cline{2-2}
 & \qquad\qquad 2x^2 - 4x + 4 \\
 & \qquad\qquad -\,(2x^2 - 4x + 4) \\
\cline{2-2}
 & \qquad\qquad\qquad\qquad 0
\end{array}
$$

STEP 6 Interpret the result of the division.

The ideal mechanical advantage for this lever can be represented by the quadratic polynomial function $m(x) = x^2 + 2x + 2$.

YOU TRY IT! #4

Using the standard long division algorithm, divide the quartic function $f(x) = x^4 + 3x^3 - 6x - 3$ by the quadratic function $g(x) = x^2 - 2$. If possible, express the quotient, $h(x)$, as a polynomial function.

See margin.

YOU TRY IT! #4 ANSWER:

Since there is a remainder, the quotient cannot be expressed as a polynomial function. However, it can be expressed as the rational function $h(x) = x^2 + 3x + 2 + \frac{1}{x^2 - 2}$.

PRACTICE/HOMEWORK

For questions 1 – 9, use long division to simplify the expression. Express any remainder as a fraction.

1. $\dfrac{5x^3 + 3x - 8}{x - 1}$

$5x^2 + 5x + 8$

2. $\dfrac{x^3 + 3x^2 + 3x + 1}{x^2 + 2x + 1}$

$x + 1$

3. $\dfrac{x^3 - 3x^2 + 3x - 1}{x^2 - 2x + 1}$

$x - 1$

4. $\dfrac{x^4 + x^2 + 1}{x^2 + x + 1}$

$x^2 - x + 1$

5. $\dfrac{x^4 + x^3 - 7x^2 - 2x + 8}{x - 2}$

$x^3 + 3x^2 - x - 4$

6. $\dfrac{x^4 - 1}{x - 1}$

$x^3 + x^2 + x + 1$

7. $\dfrac{2x^5 - 3x^4 + 4x^3 - 8x^2 + 12x - 16}{x^3 - 4}$

$2x^2 - 3x + 4$

8. $\dfrac{3x^3 - 2x^2 - 13x + 14}{x - 2}$

$3x^2 + 4x - 5 + \frac{4}{x - 2}$

9. $\dfrac{x^3 - 4x^2 - 19x + 9}{x + 3}$

$x^2 - 7x + 2 + \frac{3}{x + 3}$

For questions 10 – 12, fill in the missing parts of the synthetic division table in order to simplify each problem. Write the quotient in the space provided.

10. $\frac{6x^5 - 12x^4 + x^2 - 7x + 10}{x - 2}$

2	6	-12	0	1	-7	10
		12	0	0	2	-10
	6	0	0	1	-5	0

Quotient: $6x^4 + x - 5$

11. $\frac{9x^5 + 27x^4 + 81x + 243}{3x + 9}$

-3	9	27	0	0	81	243
		-27	0	0	0	-243
	9	0	0	0	81	0

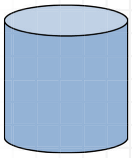
Quotient: $3x^4 + 27$

12. $\frac{2x^3 - x^2 - 23x - 20}{2x + 5}$

-2.5	2	-1	-23	-20
		-5	15	20
	2	-6	-8	0

Quotient: $x^2 - 3x - 4$

GEOMETRY



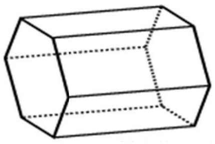
The volume of a cylinder is the product of the area of the base and the height, which is represented by the formula $V = Bh$. Use this information to answer questions 13 and 14.

13. Find the area of the base of a cylinder if the volume is represented by $f(x) = 2x^3 - 3x^2 + 7x - 3$ and the height is represented by $g(x) = 2x - 1$.
$x^2 - x + 3$

14. Find the height of the cylinder if the volume is represented by $v(x) = 6x^3 + 2x^2 + 11x - 10$ and the area of the base is represented by $b(x) = 2x^2 + 2x + 5$.
$3x - 2$

GEOMETRY

The formula for finding the volume of a prism is $V = Bh$. Use this formula to answer questions 15 and 16.



15. Find the height of a prism if the volume is represented by $V(x) = x^3 - 7x^2 + 14x - 8$ and the area of the base is represented by $B(x) = x^2 - 3x + 2$.
$x - 4$

16. Find the area of the base of a prism if the volume is represented by $V(x) = x^3 - 9x^2 + 23x - 15$ and the height is represented by $h(x) = x - 5$.
$x^2 - 4x + 3$

BUSINESS

A delivery service has a limit of 24 inches as the height of any box they will deliver. Use this situation to answer questions 17 and 18.

17. The volume of the box shown is represented by $V(x) = 4x^3 - 200x + 28$ and the area of the base of the box is represented by $B(x) = 4x^2 + 28x - 4$. Find $h(x)$, the height of the box.
$h(x) = x - 7$

18. For what value of x will result in the maximum height of a box that will be delivered by the service?
$x = 31$ inches

GEOMETRY

19. The area of a rectangle is represented by the function $f(x) = 12x^2 + 24x - 15$. What is the width of the rectangle if the length is represented by $g(x) = 2x + 5$?
$6x - 3$

BUSINESS

20. The total number of viewers who went to see *Star Wars: The Force Awakens* from Sunday to Saturday is represented by the function $v(x) = 18x^3 + 36x^2 - 8x - 6$ and the number of showings of the movie is represented by the function $s(x) = 6x + 2$, where x is the number of days since Sunday. Write a function, $a(x)$, representing the average number of viewers per showing.
$a(x) = 3x^2 + 5x - 3$