TEKS

AR.3A Compare and contrast the key attributes, including domain, range, maxima, minima, and intercepts, of a set of functions such as a set comprised of a linear, a quadratic, and an exponential function or a set comprised of an absolute value, a quadratic, and a square root function tabularly, graphically, and symbolically.

AR.7A Represent domain and range of a function using interval notation, inequalities, and set (builder) notation.

MATHEMATICAL PROCESS SPOTLIGHT

AR.1F Analyze mathematical relationships to connect and communicate mathematical ideas.

ELPS

1C Use strategic learning techniques such as concept mapping, drawing, memorizing, comparing, contrasting, and reviewing to acquire basic and grade-level vocabulary.

VOCABULARY

y-intercept, x-intercept, domain, range, maximum value, minimum value

MATERIALS

- graphing calculator

2.7 Comparing Sets of Functions

FOCUSING QUESTION How are the key attributes of different functions related?

LEARNING OUTCOMES

- I can identify and compare the domain, range, maximum or minimum values, and intercepts of a set of functions.
- I can analyze the relationships within a set of functions and communicate those relationships.

ENGAGE

The Gateway Arch in St. Louis, Missouri, is a catenary curve that can be approximated using a parabola. Which key attributes (for example, domain, range, intercepts, vertex, axis of symmetry) of a quadratic function do you see in the Arch?

See margin.

EXPLORE

In previous sections, you described the key attributes of different functions. In this section, you will use the key attributes, including domain and range, maximum or minimum values, and intercepts to compare and contrast the functions.

Use the set of functions to answer questions 1 – 7.

- $f(x) = 2(x - 5) + 8$
- $g(x) = 2(x + 5)^2 + 8$
- $h(x) = 2(5)^x + 8$

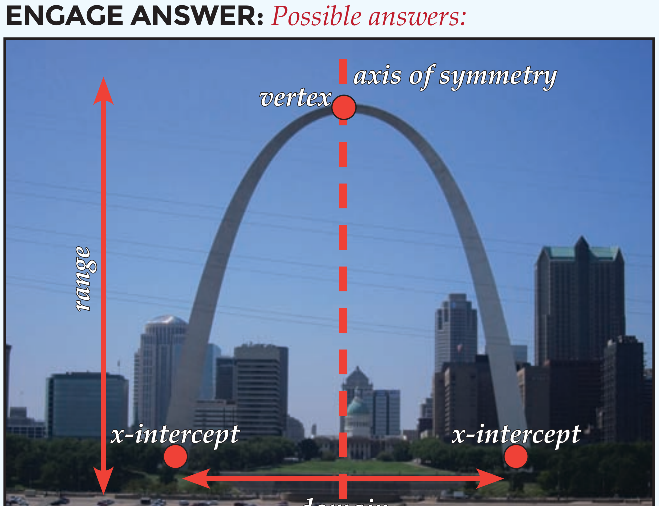
1. For each function, identify the domain and range.
See margin.

2. For each function, determine the maximum value or minimum value.
See margin.

3. For each function, determine the x-intercepts and y-intercepts.
See margin.

4. Which function(s) have a domain that includes {5}?
f(x), g(x), and h(x)

ENGAGE ANSWER: *Possible answers:*



1. *$f(x)$: domain $\{x \mid x \in \mathbb{R}\}$, range $\{f(x) \mid f(x) \in \mathbb{R}\}$*

$g(x)$: domain $\{x \mid x \in \mathbb{R}\}$, range $\{g(x) \mid g(x) \geq 8\}$

$h(x)$: domain $\{x \mid x \in \mathbb{R}\}$, range $\{h(x) \mid h(x) > 8\}$

2. *$f(x)$: no maximum or minimum value*

$g(x)$: minimum value, $g(-5) = 8$

$h(x)$: no maximum or minimum value (boundary of function values is the horizontal asymptote $y = 8$, and all function values are greater than 8)

3. *$f(x)$: x-intercept $(1, 0)$, y-intercept $(0, -2)$*

$g(x)$: no x-intercept, y-intercept $(0, 58)$

$h(x)$: no x-intercept, y-intercept $(0, 10)$

242 CHAPTER 2: ANALYZING FUNCTIONS

5. Which function(s) have a range that includes {8}?
f(x) and g(x)

6. Which function has the lower minimum value: $g(x)$ or $h(x)$? Explain your answer.
g(x) because its range includes the value eight while all the function values of h(x) are greater than eight.

7. Which function has a y-intercept closest to the x-axis?
f(x)

Use the tables that represent a subset of values for the functions p(x), q(x), and r(x) to answer questions 8 – 13.

x	$p(x)$
-4	0
-2	-1
0	-2
2	-3
4	-2
6	-1
8	0

x	$q(x)$
-1	-7.5
0	-5
1	-3.5
2	-3
3	-3.5
4	-5
5	-7.5

x	$r(x)$
-1	5
0	1
0.42	0
1	-1
2	-2
3	-2.5
4	-2.75
5	-2.875

8. For each function, identify the domain and range.
See margin.

9. For each function, determine the maximum value or minimum value.
See margin.

10. For each function, determine the x-intercepts and y-intercepts.
See margin.

11. Which function(s) have a domain that includes {0}?
p(x), q(x), and r(x)

12. Which function(s) have a range that includes {−3}?
p(x) and q(x)

13. Which function has a y-intercept closest to the x-axis?
r(x)

8. *p(x): domain $\{x \mid x \in \mathbb{R}\}$, range $\{p(x) \mid f(x) \geq -3\}$*

qx): domain $\{x \mid x \in \mathbb{R}\}$, range $\{q(x) \mid g(x) \leq -3\}$

r(x): domain $\{x \mid x \in \mathbb{R}\}$, range $\{r(x) \mid h(x) > -3\}$

9. *p(x): minimum value, $p(2) = -3$*

q(x): maximum value, $q(2) = -3$

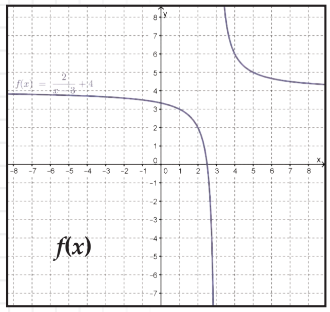
r(x): no maximum or minimum value (boundary of function values is the horizontal asymptote $y = -3$, and all function values are greater than -3)

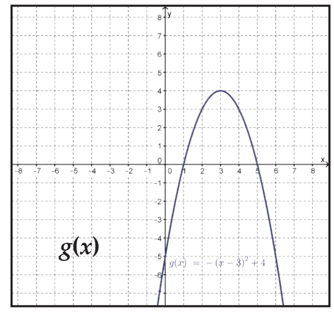
10. *p(x): x-intercepts (−4, 0) and (8, 0), y-intercept (0, −2)*

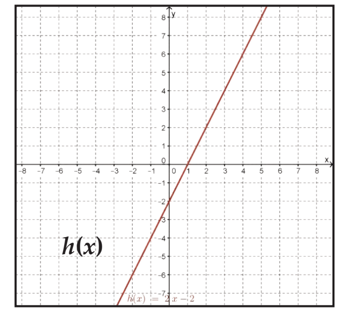
q(x): no x-intercept, y-intercept (0, −5)

r(x): x-intercept (0.42, 0), y-intercept (0, 1)

Use the graphs of the functions f(x), g(x), and h(x) below to answer questions 14 – 20.







14. For each function, identify the domain and range.
See margin.

15. For each function, determine the maximum value or minimum value.
See margin.

16. For each function, determine the x-intercepts and y-intercepts.
See margin.

17. Which function(s) have a domain that includes {3}?
g(x) and h(x)

18. Which function(s) have a range that does not include {4}?
f(x)

19. Which function does not exceed $y = 4$?
g(x)

20. Which function has a y-intercept closest to the x-axis?
h(x)

REFLECT

- **What are important characteristics of functions to look for when comparing the domain and range of a set of functions?**
  See margin.
- **What are important characteristics to look for when comparing the maximum or minimum values of a set of functions?**
  Look for range restrictions such as horizontal asymptotes or points such as a vertex.
- **What are important characteristics to look for when comparing the intercepts of a set of functions?**
  See margin.

REFLECT ANSWERS:

Many functions, including linear, quadratic, exponential, absolute value, and cubic functions have domains of all real numbers. Rational functions have vertical asymptotes that exclude values from the domain.

Many functions have range restrictions. Absolute value and quadratic functions have a vertex that represents the maximum or minimum function value. Exponential and rational functions have horizontal asymptotes.

If you are given the equation, look at the values of a, b, c, and d.

If you are given a table, look for 0 values in the x or f(x) columns.

If you are given a graph, look for the values where the graph crosses the x-axis or the y-axis.

14. *f(x): domain* $\{x \mid x \in \mathbb{R}, x \neq 3\}$, *range* $\{f(x) \mid f(x) \in \mathbb{R}, f(x) \neq 4\}$

g(x): domain $\{x \mid x \in \mathbb{R}\}$, *range* $\{g(x) \mid g(x) \leq 4\}$

h(x): domain $\{x \mid x \in \mathbb{R}\}$, *range* $\{h(x) \mid h(x) \in \mathbb{R}\}$

15. *f(x): no minimum or maximum value*

g(x): maximum value, $g(3) = 4$

h(x): no maximum or minimum value

16. *f(x): x-intercept (2.5, 0), y-intercept* $(0, 3\frac{1}{3})$

g(x): x-intercepts (1, 0) and (5, 0), y-intercept (0, –5)

h(x): x-intercept (1, 0), y-intercept (0, –2)

244 CHAPTER 2: ANALYZING FUNCTIONS

EXPLAIN

You can use graphs, tables, or equations to compare and contrast the key attributes of a set of functions. The chart below summarizes ways to determine the domain, range, maximum or minimum value, and intercepts of six types of functions.

FUNCTION	DOMAIN	RANGE	MAXIMUM OR MINIMUM	x-INTERCEPT	y-INTERCEPT
LINEAR $f(x) = a(bx - c) + d$	$\{x \mid x \in \mathbb{R}\}$ $(-\infty, \infty)$ $-\infty < x < \infty$	$\{f(x) \mid f(x) \in \mathbb{R}\}$ $(-\infty, \infty)$ $-\infty < f(x) < \infty$	NONE	$\left(\frac{ac - d}{ab}, 0\right)$	$(0, -ac + d)$
ABSOLUTE VALUE $f(x) = a\lvert bx - c\rvert + d$	$\{x \mid x \in \mathbb{R}\}$ $(-\infty, \infty)$ $-\infty < x < \infty$	VERTEX IS $(\frac{c}{b}, d)$ ■ If $a > 0$, $\{f(x) \mid f(x) \geq d\}$ $(-\infty, d]$ $d \leq f(x) < \infty$ ■ If $a < 0$, $\{f(x) \mid f(x) \leq d\}$ $[d, \infty)$ $-\infty < f(x) \leq d$	VERTEX IS $(\frac{c}{b}, d)$ ■ If $a > 0$, minimum value $y = d$ ■ If $a < 0$, maximum value $y = d$	$\left(\frac{c \pm \frac{d}{a}}{b}, 0\right)$	$(0, a\lvert c\rvert + d)$
QUADRATIC $f(x) = a(bx - c)^2 + d$	$\{x \mid x \in \mathbb{R}\}$ $(-\infty, \infty)$ $-\infty < x < \infty$	VERTEX IS $(\frac{c}{b}, d)$ ■ If $a > 0$, $\{f(x) \mid f(x) \geq d\}$ $[d, \infty)$ $d \leq f(x) < \infty$ ■ If $a < 0$, $\{f(x) \mid f(x) \leq d\}$ $(-\infty, d]$ $-\infty < f(x) \leq d$	VERTEX IS $(\frac{c}{b}, d)$ ■ If $a > 0$, minimum value $y = d$ ■ If $a < 0$, maximum value $y = d$	$\left(\frac{c \pm \sqrt{\frac{-d}{a}}}{b}, 0\right)$	$(0, ac^2 + d)$
CUBIC $f(x) = a(bx - c)^3 + d$	$\{x \mid x \in \mathbb{R}\}$ $(-\infty, \infty)$ $-\infty < x < \infty$	$\{f(x) \mid f(x) \in \mathbb{R}\}$ $(-\infty, \infty)$ $-\infty < f(x) < \infty$	NONE	UP TO 3	$(0, -ac^3 + d)$
EXPONENTIAL $f(x) = a(b)^{kx - c} + d$	$\{x \mid x \in \mathbb{R}\}$ $(-\infty, \infty)$ $-\infty < x < \infty$	HORIZONTAL ASYMPTOTE IS $f(x) = d$ ■ If $a > 0$, $\{f(x) \mid f(x) > d\}$ (d, ∞) $d < f(x) < \infty$ ■ If $a < 0$, $\{f(x) \mid f(x) < d\}$ $(-\infty, d)$ $-\infty < f(x) < d$	NONE	0 OR 1	$(0, \frac{a}{b^c} + d)$
RATIONAL $f(x) = \frac{a}{bx - c} + d$	VERTICAL ASYMPTOTE IS $x = \frac{c}{b}$ $\{x \mid x \in \mathbb{R}, x \neq \frac{c}{b}\}$ $(-\infty, \frac{c}{b})$ OR $(\frac{c}{b}, \infty)$ $x \neq \frac{c}{b}$	HORIZONTAL ASYMPTOTE IS $f(x) = d$ $\{f(x) \mid f(x) \in \mathbb{R}, f(x) \neq d\}$ $(-\infty, d)$ OR (d, ∞) $f(x) \neq d$	NONE	$\left(\frac{cd - a}{bd}, 0\right)$	$(0, \frac{a}{-c} + d)$

INSTRUCTIONAL HINTS

Students might benefit from having this table on a piece of paper beside them to use as a reference chart as they practice this section. Wean students off of it as they become more confident.

For students who are struggling with memorizing the formulas for x- and y-intercepts, remind them that they can always substitute 0 for x and y and solve for the desired variable.

INTEGRATING TECHNOLOGY

For the key attributes of different types of functions, use a graphing calculator display or online software to show students how the domain and range, maximum or minimum values, and intercepts change as the parameters are transformed. Encourage students to look for similarities among different types of functions.

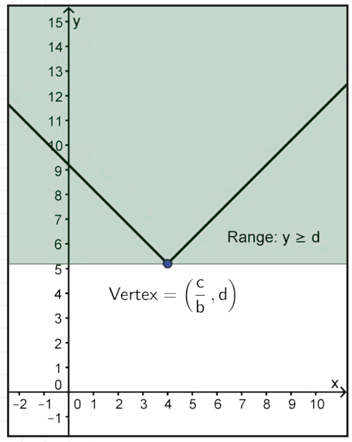
DOMAIN AND RANGE

Watch Explain and You Try It Videos

or click here

For many functions, the domain and range are all real numbers. However, certain function types have restrictions that prevent some numbers from being used in the set of input values, or domain. Some functions include operations that restrict the set of output values, or range.

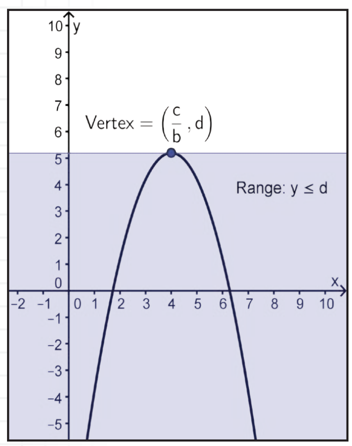
Absolute Value Functions



Absolute value functions include the operation of taking the absolute value, which always generates a non-negative output. Hence, the range of the parent function is restricted to non-negative numbers, $y \geq 0$. As the function is vertically translated, so is the range restriction.

Remember that if $a > 0$, then the graph opens upward and the range is greater than or equal to the y-value of the vertex. If $a < 0$, then the graph opens downward and the range is less than or equal to the y-value of the vertex.

Quadratic Functions



Quadratic functions have a similar range restriction as absolute value functions. Quadratic functions include the operation of squaring, which always generates a non-negative output. Hence, the range of the parent function is restricted to non-negative numbers, $y \geq 0$. As the function is vertically translated, so is the range restriction.

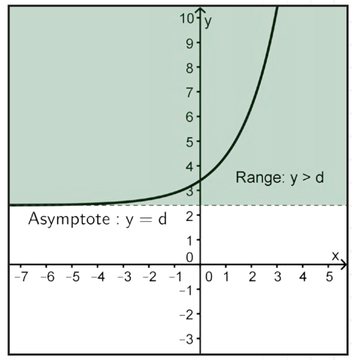
Remember that if $a > 0$, then the graph opens upward and the range is greater than or equal to the y-value of the vertex. If $a < 0$, then the graph opens downward and the range is less than or equal to the y-value of the vertex.

Exponential Functions

Exponential functions involve raising a base, b, to a power that includes the variable, x. The graph of an exponential function includes all of the function values generated by raising a base to a particular power. There is no restriction on the domain. However, when you raise a given number to a power, there is a limit for the resulting function value. For exponential growth, that limit occurs when the value of x decreases and approaches $-\infty$. For exponential decay, that limit occurs when the value of x increases and approaches ∞.

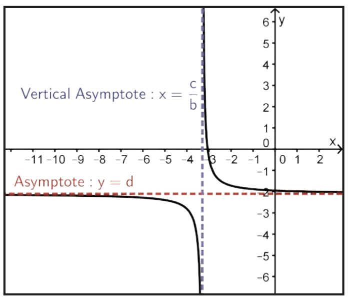
246 CHAPTER 2: ANALYZING FUNCTIONS

Hence, the range of the parent function is restricted to values greater than, but not equal to, the y-value of the horizontal asymptote, $y > d$. As the function is vertically translated, so is the asymptote and the range restriction.



Remember that if $a > 0$, then the graph grows or decays above the asymptote and the range is greater than the y-value of the asymptote. If $a < 0$, then the graph grows or decays below the asymptote and the range is less than the y-value of the asymptote.

Rational Functions



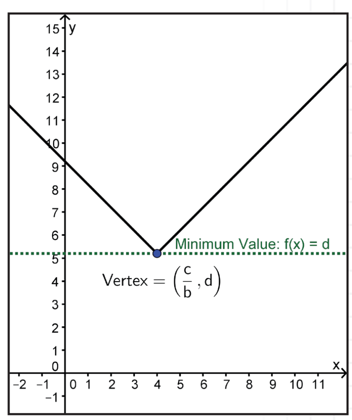
Rational functions involve a ratio with x in the denominator. Division by zero is undefined, so the denominator can never equal 0. Thus, there are both domain and range restrictions for rational functions.

For example, in the ratio $\frac{3}{x-4}$, if $x - 4 = 0$, then the ratio is not a real number. So, if $x = 4$, then there is no real number value for that ratio and $x = 4$ cannot be included in the domain of a function using that ratio. As x gets very large or very small, the absolute value of the denominator becomes a very large number and the ratio is very close to 0. For the parent function $f(x) = \frac{1}{x}$, this generates a horizontal asymptote at $y = 0$. When the parent function is vertically translated, so is the horizontal asymptote.

Rational functions have a vertical asymptote at $x = \frac{c}{b}$ and a horizontal asymptote at $y = d$. The value of the vertical asymptote, $x = \frac{c}{b}$, is excluded from the domain and the value of the horizontal asymptote, $y = d$, is excluded from the range.

MAXIMUM OR MINIMUM VALUES

For many functions, the range is boundless and there is not a maximum or minimum value. However, if you consider a subset of the domain of a function, you can generate what is known as a **local maximum** or **local minimum** value. In other words, the value isn't an absolute maximum or minimum for the entire function, but it is a maximum or minimum value for a local region of the graph.



Absolute Value Functions

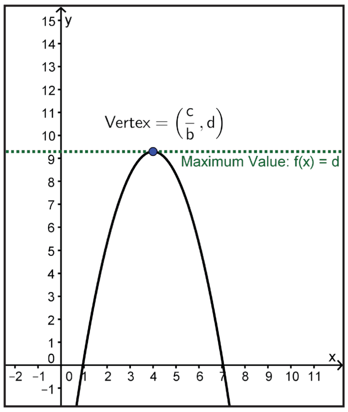
Absolute value functions have a maximum or minimum function value for the entire function at the vertex. If $a > 0$, then the y-coordinate of the vertex is a minimum value for the function. If $a < 0$, then the y-coordinate of the vertex is a maximum value for the function.

STRATEGIES FOR SUCCESS

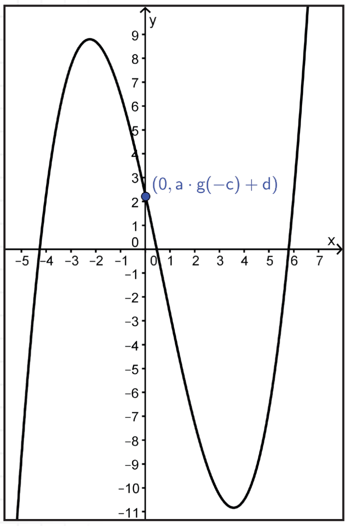
Students can create a foldable graphic organizer to summarize the key attributes of different types of functions. Use the categories of domain and range, maximum and minimum values, intercepts, and other properties to create a four-shutter foldable graphic organizer. Students can affix completed organizers into their interactive mathematics notebooks or math journals.

Quadratic Functions

Quadratic functions also have a maximum or minimum function value for the entire function at the vertex. If $a > 0$, then the y-coordinate of the vertex is a minimum value for the function. If $a < 0$, then the y-coordinate of the vertex is a maximum value for the function.



y-INTERCEPTS



Most functions have a y-intercept, $(0, g(0))$, which is the function value when $x = 0$. Graphically, the y-intercept is where the graph crosses or touches the y-axis. From the equation, the y-intercept can be calculated by letting $x = 0$ and using the order of operations to simplify the expression. If the equation is in general form with the parameters a, b, c, and d, the y-intercept is affected only by a, c, and d.

For any function, $g(x)$, that is a transformation of the parent function $f(x)$, the y-intercept can be calculated as shown.

$$g(x) = af(bx - c) + d$$
$$g(0) = af(b(0) - c) + d$$
$$g(0) = af(-c) + d$$

x-INTERCEPTS

The x-intercepts of a function are the points where the graph touches or crosses the x-axis, or points where the y-values or function values are equal to 0. x-intercepts can be determined from the graph, locating points $(x, 0)$ in a table of values, or by setting $f(x) = 0$ and solving the related equation for x.

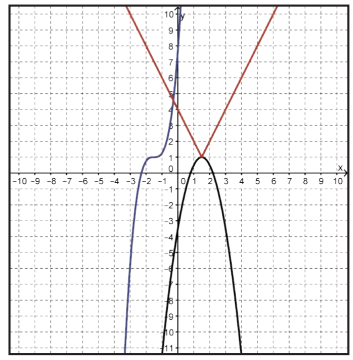
COMPARING KEY ATTRIBUTES OF FUNCTIONS

- Maximum or minimum values, such as a vertex or horizontal asymptote, help identify range restrictions.
- Horizontal or vertical asymptotes that split a graph into different pieces (e.g., rational functions) identify values that must be excluded from the domain or range of the function.
- Intercepts are found where the graph touches or crosses the x-axis or y-axis.

248 CHAPTER 2: ANALYZING FUNCTIONS

EXAMPLE 1

Compare the domain and range of $f(x) = 2(x + 1.5)^3 + 1$, $g(x) = -2(x - 1.5)^2 + 1$, and $h(x) = 2|x - 1.5| + 1$.



STEP 1 **Identify the graph of each function.**

The graph in blue represents a cubic function, $f(x) = 2(x + 1.5)^3 + 1$. The graph in black represents a quadratic function, $g(x) = -2(x - 1.5)^2 + 1$. The graph in red represents an absolute value function, $h(x) = 2|x - 1.5| + 1$.

STEP 2 **Identify the domain and range of the cubic function, $f(x) = 2(x + 1.5)^3 + 1$.**

The domain contains all real numbers, $\{x \mid x \in \mathbb{R}\}$. The range also contains all real numbers, $\{y \mid y \in \mathbb{R}\}$

STEP 3 **Identify the domain and range of the quadratic function, $g(x) = -2(x - 1.5)^2 + 1$.**

The domain contains all real numbers, $\{x \mid x \in \mathbb{R}\}$. Since $-2 < 0$, the graph opens downward, and its vertex is at $\left(\frac{c}{b}, d\right)$, which is (1.5, 1), the range is limited to real numbers less than or equal to 1, $\{g(x) \mid g(x) \leq 1\}$.

STEP 4 **Identify the domain and range of the absolute value function, $h(x) = 2|x - 1.5| + 1$.**

The domain contains all real numbers, $\{x \mid x \in \mathbb{R}\}$. Since $2 > 0$, the graph opens upwards, and its vertex is at $\left(\frac{c}{b}, d\right)$, which is (1.5, 1), the range is limited to real numbers greater than or equal to 1, $\{h(x) \mid h(x) \geq 1\}$.

By identifying the a, b, c, and d of the equations of the functions and the types of functions they represent, and looking at their graphs, we can see that all three functions have domains containing all real numbers. The cubic function, f(x), has a range including all real numbers, but the values of a and d in the quadratic function, g(x), limit its range to $\{g(x) \mid g(x) \leq 1\}$, and in the absolute value function, h(x), to $\{h(x) \mid h(x) \geq 1\}$.

ADDITIONAL EXAMPLE

Compare the domain and range of the functions below.

$p(x) = 4(0.5x - 3)^3 - 9$

$q(x) = 4^{(0.5x - 3)} - 9$

$r(x) = \frac{4}{0.5x - 3} - 9$

The domain of the cubic function contains all real numbers, $\{x \mid x \in \mathbb{R}\}$. The range also contains all real numbers, $\{p(x) \mid p(x) \in \mathbb{R}\}$.

The domain of the exponential function contains all real numbers, $\{x \mid x \in \mathbb{R}\}$. The range is restricted by the horizontal asymptote $q(x) = -9$, $\{q(x) \mid q(x) > -9\}$.

The domain of the rational function contains all real numbers except the vertical asymptote at $x=6$, $\{x \mid x \in \mathbb{R}, x \neq 6\}$. The range of the rational function contains all real numbers except the horizontal asymptote at $r(x) = -9$, $\{r(x) \mid r(x) \in \mathbb{R}, r(x) \neq -9\}$.

YOU TRY IT! #1 ANSWER:

The domain of the linear function, $f(x)=\frac{3}{4}x+4.8$, and the exponential function, $g(x)=\left(\frac{3}{4}\right)^x+4.8$, contain all real numbers. The domain for both is $\{x \mid x \in \mathbb{R}\}$. The domain of the rational function, $h(x)=\frac{0.75}{x}+4.8$, excludes the value of the vertical asymptote, $x = 0$.

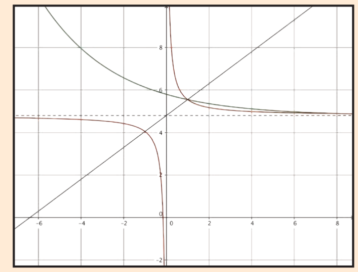
Written in set builder notation, $\{x \mid x, x \neq 0\}$.

The range of the linear function, $f(x)=\frac{3}{4}x+4.8$, contains all real numbers, $\{f(x) \mid f(x) \in \mathbb{R}\}$. The range of the exponential function, $g(x)=\left(\frac{3}{4}\right)^x+4.8$, is bounded by the value of its horizontal asymptote, $y = 4.8$. Written in set builder notation, this is $\{g(x) \mid g(x) > 4.8\}$. The range of the rational function, $h(x)=\frac{0}{x}+4.8$, excludes the value of the horizontal asymptote, $y = 4.8$. Written in set builder notation, $\{h(x) \mid h(x) \neq 4.8\}$.

YOU TRY IT! #1

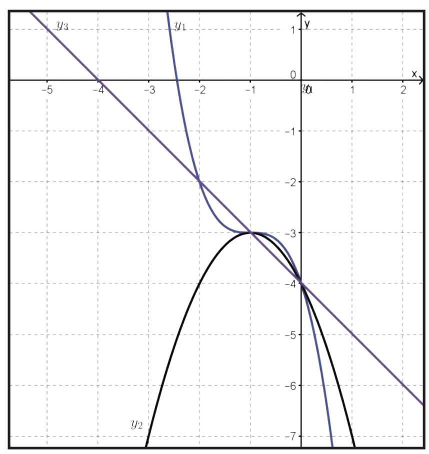
Compare the domain and range of $f(x) = \frac{3}{4}x + 4.8$, $g(x) = \left(\frac{3}{4}\right)^x + 4.8$, and $h(x) = \frac{0.75}{x} + 4.8$.

See margin.



EXAMPLE 2

Compare the intercepts of the functions y_1, y_2, and y_3 by analyzing their graphs and tables.



x	y_1	y_2	y_3
-5	1	-19	61
-4	0	-12	24
-3	-1	-7	5
-2	-2	-4	-2
-1	-3	-3	-3
0	-4	-4	-4
1	-5	-7	-11
2	-6	-12	-30
3	-7	-19	-67

STEP 1 **Determine the x-intercepts of y_1, y_2, and y_3.**

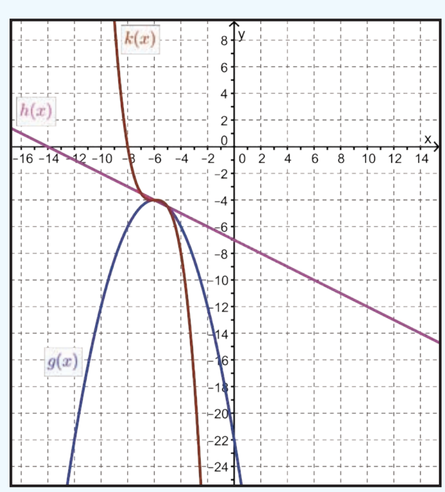
Looking at the table for coordinate pairs with $y = 0$, y_1 has an x-intercept at (-4, 0). This is verified by the graph of y_1.

The table for y_2 doesn't show a coordinate pairs with $y = 0$. Moreover, the graph of y_2 doesn't show an x-intercept.

ADDITIONAL EXAMPLE

Compare the intercepts of the functions $h(x)$, $g(x)$, and $k(x)$ by analyzing their graphs and tables.

x	-14	-12	-10	-8	-6	-4	-2	0	2
$h(x)$	0	-1	-2	-3	-4	-5	-6	-7	-8
$g(x)$	-36	-22	-12	-6	-4	-6	-12	-22	-36
$k(x)$	252	104	28	0	-4	-8	-36	-112	-260



$h(x)$ has an x-intercept at (-14, 0) and a y-intercept at (0, -7). $g(x)$ does not have an x-intercept but has a y-intercept at (0, -22). $k(x)$ has an x-intercept at (-8, 0) and a y-intercept at (0, -112).

250 CHAPTER 2: ANALYZING FUNCTIONS

Likewise, the table for y_3 doesn't show a coordinate pairs with $y = 0$. However, the table shows a change of signs for the y-values from $x = -3$ and $x = -2$. The graph confirms an x-intercept around (-2.5, 0).

STEP 2 **Determine the y-intercepts of y_1, y_2, and y_3.**

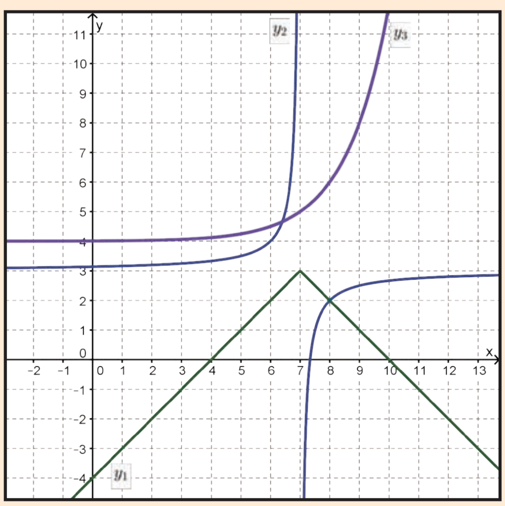
Looking at the table for y-values where $x = 0$, all three functions have coordinate pairs at (0, -4). The graphs of all three functions verify a y-intercept at (0, -4).

STEP 3 **Compare the x- and y-intercepts for y_1, y_2, and y_3.**

All three functions have a y-intercept at (0, -4). The functions y_1 and y_3 have x-intercepts at (-4, 0) and (-2.5, 0) and y_2 has no x-intercept.

YOU TRY IT! #2

Compare the intercepts of the functions y_1, y_2, and y_3 by analyzing their graphs and tables.



x	y_1	y_2	y_3
-2	-6	3.1111	4.002
-1	-5	3.125	4.0039
0	-4	3.1429	4.0078
1	-3	3.1667	4.0156
2	-2	3.2	4.0313
3	-1	3.25	4.0625
4	0	3.3333	4.125
5	1	3.5	4.25
6	2	4	4.5
7	3	ERROR	5
8	2	2	6
9	1	2.5	8
10	0	2.6667	12
11	-1	2.75	20

See margin.

YOU TRY IT! #2 ANSWER:

The x-intercepts of y_1 are (4, 0) and (10, 0). The x-intercept of y_2 is a little to the right of (7, 0). There doesn't appear to be an x-intercept for y_3.

The y-intercept of y_1 is (0, -4). The y-intercept of y_2 is a little above (0, 3), and the y-intercept of y_3 is a little above (0, 4).

According to the table, the y-intercept of y_2 is (0, 3.1429), and the y-intercept of y_3 is (0, 4.0078).

ADDITIONAL EXAMPLE

Compare the maximum and minimum values of

$y_1 = -0.75|2x + 3| - 4$,

$y_2 = \frac{0.75}{2x + 3} - 4$,

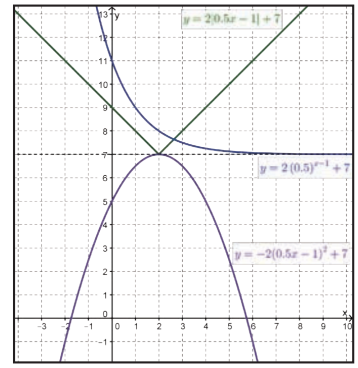
$y_3 = 0.75(2x + 3)^2 - 4$.

y_1 and y_3 have vertices at $(-\frac{3}{2}, -4)$. This is the maximum value for y_1, the absolute value function. For y_3, the quadratic function, this is the minimum value.

y_2, the rational function, does not have a minimum or maximum value, but it is restricted by the horizontal asymptote at $y = -4$ and the vertical asymptote at $x = -\frac{3}{2}$.

EXAMPLE 3

Compare the maximum and minimum values of $y = 2|0.5x - 1| + 7$, $y = 2(0.5)^{x-1} + 7$, and $y = -2(0.5x - 1)^2 + 7$.



STEP 1 **Determine the maximum or minimum values of the absolute value function $y = 2|0.5x - 1| + 7$.**

The vertex of the function is where $x = \frac{c}{b} = \frac{1}{0.5} = 2$ and $y = d = 7$. Since $a = 2$, which is greater than zero, the function value, 7, at the vertex, (2, 7) is the minimum value.

STEP 2 **Determine the maximum or minimum values of the exponential function $y = 2(0.5)^{x-1} + 7$.**

For $y = 2(0.5)^{x-1} + 7$, $b = 0.5$, and so it is an exponential decay function. At $y = d$, which is 7, there is a horizontal asymptote. Although the y-values become smaller and closer to $y = 7$ as the x-values increase, there isn't a minimum value. If the function value were equal to 7,

$$7 = 2(0.5)^{x-1} + 7$$

$$7 - 7 = 2(0.5)^{x-1} + 7 - 7$$

$$\frac{0}{2} = \frac{2(0.5)^{x-1}}{2}$$

$$0 = (0.5)^{x-1}$$

But there is no value of x for which (0.5) can be raised to equal 0.

STEP 3 **Determine the maximum or minimum values of the quadratic function $y = -2(0.5x - 1)^2 + 7$.**

The vertex of the function is where $x = \frac{c}{b} = \frac{1}{0.5} = 2$ and $y = d = 7$. Since $a = -2$, which is less than zero, the function value, 7, at the vertex, (2, 7) is the maximum value.

STEP 4 **Compare the maximum or minimum for these functions.**

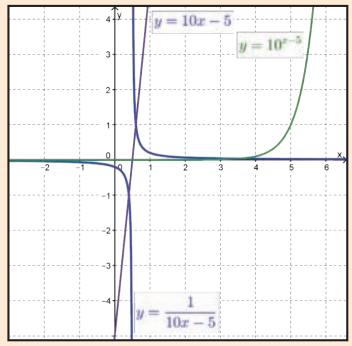
Both $y = 2|0.5x - 1| + 7$ and $y = -2(0.5x - 1)^2 + 7$ have vertices at (2, 7). This is the minimum value for the absolute value function and the maximum value for the quadratic function. The exponential function has no minimum value but is restricted by the horizontal asymptote at $y = 7$.

252 CHAPTER 2: ANALYZING FUNCTIONS

YOU TRY IT! #3

Compare the maximum and minimum values of $y = 10^{x-5}$, $y = \frac{1}{10x-5}$, and $y = 10x - 5$.

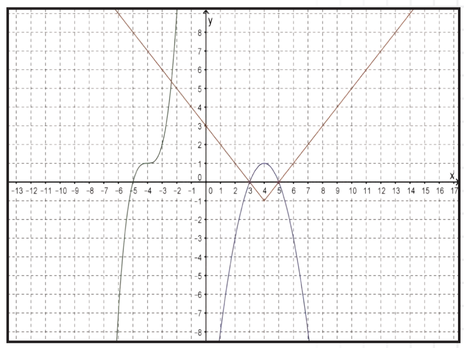
See margin.



PRACTICE/HOMEWORK

Use the functions below to answer questions 1 – 5.

$$f(x) = 8(\tfrac{1}{2}x + 2)^3 + 1$$
$$g(x) = -4(\tfrac{1}{2}x - 2)^2 + 1$$
$$h(x) = 2\left|\tfrac{1}{2}x - 2\right| - 1$$



1. Identify and compare the domain of the functions.
See margin.

2. Identify and compare the range of the functions.
See margin.

3. Identify and compare the *x*-intercepts of the functions.
See margin.

4. Identify and compare the *y*-intercepts of the functions.
See margin.

5. Identify and compare the maximum and minimum values of the functions.
See margin.

YOU TRY IT! #3 ANSWER:

None of these functions have a minimum or maximum value. However, the exponential function $y = 10^{x-5}$ has a restriction on its y-values, $y > 0$, which is its horizontal asymptote. But there is not a minimum value since, as x-values decrease, y-values continue to decrease, getting closer to zero.

1. *f(x): domain $\{x \mid x \in \mathbb{R}\}$*

g(x): domain $\{x \mid x \in \mathbb{R}\}$

h(x): domain $\{x \mid x \in \mathbb{R}\}$

All three functions have the same domain, all real numbers.

2. *f(x): range $\{f(x) \mid f(x) \in \mathbb{R}\}$*

g(x): range $\{g(x) \mid g(x) \le 1\}$

h(x): range $\{h(x) \mid h(x) \ge -1\}$

The cubic function, f(x), has a range including all real numbers, but the values of a and d in the quadratic function, g(x), limit its range to $\{g(x) \mid g(x) \le 1\}$, and in the absolute value function, h(x), to $\{h(x) \mid h(x) \ge -1\}$.

3. *f(x): x-intercept (-5, 0)*

g(x): x-intercept (3, 0) and (5, 0)

h(x): x-intercept (3, 0) and (5, 0)

There is only one x-intercept for the cubic function, f(x), which is (-5, 0). The quadratic function, g(x), and the absolute value function, h(x), have the same x-intercepts, (3, 0) and (5, 0).

4. *f(x): y-intercept (0, 65)*

g(x): y-intercept (0, -15)

h(x): y-intercept (0, 3)

The y-intercept of the cubic function, f(x), is (0, 65) and crosses the y-axis at the highest point out of the three functions. The y-intercept of the quadratic function, g(x), is (0, -15) and crosses the y-axis at the lowest point of the three functions. The y-intercept of the absolute value function, h(x), is (0, 3) and crosses the y-axis in the middle of the three functions.

5. *f(x): no maximum or minimum value*

g(x): maximum value, g(4) = 1

h(x): minimum value, h(4) = -1

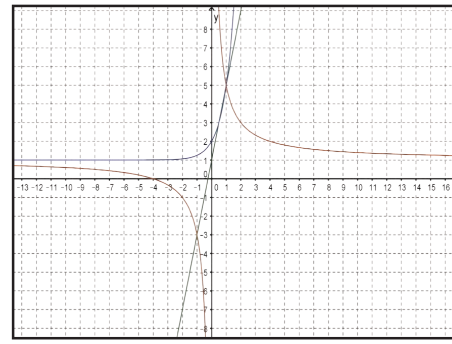
The cubic function, f(x), does not have a minimum or maximum value. The quadratic function, g(x), has a vertex at (4, 1). This point represents the maximum value of the function. The absolute value function, h(x), has a vertex at (4, -1). This point represents the minimum value of the function.

Use the functions below to answer questions 6 – 10.

$p(x) = 4x + 1$

$q(x) = (4)^x + 1$

$r(x) = \frac{4}{x} + 1$



6. Identify and compare the domain of the functions.
See margin.

7. Identify and compare the range of the functions.
See margin.

8. Identify and compare the x-intercepts of the functions.
See margin.

9. Identify and compare the y-intercepts of the functions.
See margin.

10. Identify and compare the maximum and minimum values of the functions.
See margin.

Use the functions below to answer questions 11 – 15.

$f(x) = 4(x + 4) + 8$

$g(x) = -2(x + 3)^2 + 8$

$h(x) = (x + 3)^3 + 8$

x	$f(x)$	$g(x)$	$h(x)$
-6	0	-10	-19
-5	4	0	0
-4	8	6	7
-3	12	8	8
-2	16	6	9
-1	20	0	16
0	24	-10	35
1	28	-24	72

11. Identify and compare the domain of the functions.
See margin.

12. Compare the range of the functions.
See margin.

13. Identify and compare the x-intercepts of the functions.
See margin.

14. Identify and compare the y-intercepts of the functions.
See margin.

15. Identify and compare the maximum and minimum values of the functions.
See margin.

6. *$p(x)$: domain $\{x \mid x \in \mathbb{R}\}$*

$q(x)$: domain $\{x \mid x \in \mathbb{R}\}$

$r(x)$: domain $\{x \mid x \in \mathbb{R}, x \neq 0\}$.

The domain of the linear function, $p(x)$, and the exponential function, $g(x)$, contain all real numbers. The domain for both is $\{x \mid x \in \mathbb{R}\}$. The domain of the rational function, $r(x)$, excludes the value of the vertical asymptote, $x = 0$, so its domain is a subset of the domain of $p(x)$ or $q(x)$.

7. *$p(x)$: range $\{p(x) \mid p(x) \in \mathbb{R}\}$*

$q(x)$: range $\{q(x) \mid q(x) > 1\}$

$r(x)$: range $\{r(x) \mid r(x) \in \mathbb{R}, y \neq 1\}$

The range of the linear function, $p(x)$, contains all real numbers, $\{p(x) \mid p(x) \in \mathbb{R}\}$. The range of the exponential function, $q(x)$, is bounded by the value of its horizontal asymptote, $y = 1$, $\{q(x) \mid q(x) > 1\}$. The range of the rational function, $r(x)$, excludes the value of the horizontal asymptote, $y = 1$, $\{r(x) \mid r(x) \in \mathbb{R}, y \neq 1\}$. The range of $r(x)$ is a subset of the range of $p(x)$ and the range of $q(x)$ is a subset of the ranges of both $p(x)$ and $r(x)$.

8. *$p(x)$: x-intercept $(-0.25, 0)$*

$q(x)$: no x-intercept

$r(x)$: x-intercept $(-4, 0)$

There is only one x-intercept for the linear function, $p(x)$ and the rational function $r(x)$. The exponential function, $q(x)$, does not have an x-intercept. The x-coordinate of the x-intercept of $p(x)$ is greater than the x-coordinate of the x-intercept of $r(x)$.

9. *$p(x)$: y-intercept $(0, 1)$*

$q(x)$: y-intercept $(0, 2)$

$r(x)$: no y-intercept

The y-intercept of the linear function $p(x)$, is $(0, 1)$ and crosses the y-axis at the highest point out of the three function slightly below the y-intercept of the exponential function, $q(x)$, which is $(0, 2)$. The rational function, $r(x)$ does not cross the y-axis so it does not have a y-intercept.

10. *$p(x)$: no maximum or minimum value*

$q(x)$: no maximum or minimum value

$h(x)$: no maximum or minimum value

None of these functions have a minimum or maximum value.

11-15. *See page 255*

Use the functions below to answer questions 16 – 20.

$$p(x) = \frac{1}{4}(x + 8)^2 - 4$$
$$q(x) = \frac{1}{2}|x + 4| - 8$$
$$r(x) = 2^{(x+1)} - 4$$

x	$p(x)$	$q(x)$	$r(x)$
-5	-1.75	-7.5	-3.938
-4	0	-8	-3.875
-3	2.25	-7.5	-3.75
-2	5	-7	-3.5
-1	8.25	-6.5	-3
0	12	-6	-2
1	16.25	-5.5	0
2	21	-5	4

16. Identify and compare the domain of the functions.
See margin.

17. Identify and compare the range of the functions.
See margin.

18. Identify and compare the x-intercepts of the functions.
See margin.

19. Identify and compare the y-intercepts of the functions.
See margin.

20. Identify and compare the maximum and minimum values of the functions.
See margin.

11. *f(x): domain* $\{x \mid x \in \mathbb{R}\}$

g(x): domain $\{x \mid x \in \mathbb{R}\}$

h(x): domain $\{x \mid x \in \mathbb{R}\}$

By identifying the a, b, c, and d of the equations of the functions and the types of functions they represent, and looking at their graphs, we can see that all three functions have domains containing all real numbers.

12. *f(x): range* $\{f(x) \mid f(x) \in \mathbb{R}\}$

g(x): range $\{g(x) \mid g(x) \leq 8\}$

h(x): range $\{h(x) \mid h(x) \in \mathbb{R}\}$

The linear function, f(x), and the cubic function, h(x), have a range including all real numbers, but the values of a and d in the quadratic function, g(x), limit its range to $\{y \mid y \leq 8\}$*. The range of g(x) is a subset of the ranges of both f(x) and h(x).*

13. *f(x): x-intercept (-6, 0)*

g(x): x-intercepts (-5, 0) and (−1, 0)

h(x): x-intercept (-5, 0)

The linear function, f(x), has just one x-intercept while the quadratic function, g(x), has two intercepts. The cubic function, h(x), has only one x-intercept because its inflection point is above the x-axis. The functions g(x) and h(x) share a common x-intercept, (−5, 0).

14. *f(x): y-intercept (0, 24)*

g(x): y-intercept (0, −10)

h(x): y-intercept (0, 35)

All three functions have only one y-intercept. Although the equations are similar, their y-intercepts are different because the parameters a, b, c, and d affect the y-intercepts of linear, quadratic, and cubic functions differently.

15. *f(x): no maximum or minimum value*

g(x): maximum value, g(-3) = 8

h(x): no maximum or minimum value

The linear function, f(x), and the cubic function, h(x), do not have a minimum or maximum value. The quadratic function, g(x), has a vertex at (-3, 8). This point represents the maximum value of the function.

16. *p(x): domain* $\{x \mid x \in \mathbb{R}\}$

q(x): domain $\{x \mid x \in \mathbb{R}\}$

r(x): domain $\{x \mid x \in \mathbb{R}\}$.

By identifying the a, b, c, and d of the equations of the functions and the types of functions they represent, and looking at their graphs, we can see that all three functions have domains containing all real numbers.

17. *p(x): range* $\{p(x) \mid p(x) \geq -4\}$

q(x): range $\{q(x) \mid q(x) \geq -8\}$

r(x): range $\{r(x) \mid r(x) > -4\}$

The values of a and d in the quadratic function, p(x), limit its range to $\{p(x) \mid p(x) \geq -4\}$ *and in the absolute value function, q(x), to* $\{q(x) \mid q(x) \geq -8\}$*. The range of the exponential function, r(x), is bounded by the value of its horizontal asymptote, y = -4 to* $\{r(x) \mid r(x) > -4\}$*.*

18. *p(x): x-intercept (-12, 0) and (-4, 0)*

q(x): x-intercept (-20, 0) and (12, 0)

r(x): x-intercept (1, 0)

There are two x-intercepts for the quadratic function, p(x), and the absolute value function, q(x). The exponential function, r(x), has one x-intercept.

19. *p(x): y-intercept (0, 12)*

q(x): y-intercept (0, -6)

r(x): y-intercept (0, -2)

All three functions have only one y-intercept. The quadratic function, p(x), has a y-intercept above the x-axis and the absolute value function, q(x), and the exponential function, r(x), have y-intercepts below the x-axis.

20. *p(x): minimum value, p(-8) = -4*

q(x): minimum value, p(-4) = -8

r(x): no maximum or minimum value

The quadratic function, p(x), has a vertex at (-8, -4). This point represents the minimum value of the function. The absolute value function, q(x), has a vertex at (-4, -8). This point represents the minimum value of the function. The exponential function, r(x), does not have a minimum or maximum value.