Transforming and Analyzing Exponential Functions

2.6

FOCUSING QUESTION. How do transformations affect the domain, range, intercepts, and maximum/minimum values of exponential functions?

LEARNING OUTCOMES

- I can compare and contrast the key attributes of an exponential function with polynomial, rational, or absolute value functions when I have the function as a table, graph, or written symbolically.
- I can represent the domain and range of an exponential function in a variety of ways, including interval notation, inequalities, and set builder notation.
- I can use multiple representations, including symbols, graphs, tables, and language to communicate mathematical ideas.

ENGAGE

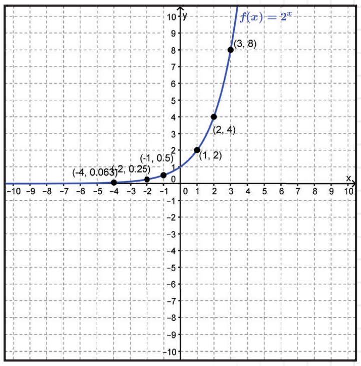
Marie Curie studied radioactive elements, developing the theory of radioactivity and earning a Nobel Prize in Physics. Radioactive substances decay at a steady rate. The half-life of a substance is the amount of time it takes for half of the substance to decay. For example, the half-life of polonium-209 is 102 years. If Professor Curie worked in 1918 with a 50 gram sample of polonium-209, how many grams would remain in 2122?

12.5 grams

Marie Curie, Christie's, Wikimedia Commons

EXPLORE

A general form of an exponential function is $f(x) = a(b)^{kx-c} + d$, where a, b, k, c, and d all represent real numbers and b additionally represents the base of the exponential function. The graph of an exponential growth parent function, using $b = 2$, is shown.



You have seen how the parameters a, b, c, and d have affected other types of functions. In this section, to avoid confusion, we will use k to represent the transformation parameter b since the constant b represents the base of the exponential function.

TEKS

AR.3A Compare and contrast the key attributes, including domain, range, maxima, minima, and intercepts, of a set of functions such as a set comprised of a linear, a quadratic, and an exponential function or a set comprised of an absolute value, a quadratic, and a square root function tabularly, graphically, and symbolically.

AR.7A Represent domain and range of a function using interval notation, inequalities, and set (builder) notation.

MATHEMATICAL PROCESS SPOTLIGHT

AR.1D Communicate mathematical ideas, reasoning, and their implications using multiple representations, including symbols, diagrams, graphs, and language as appropriate.

ELPS

1E Internalize new basic and academic language by using and reusing it in meaningful ways in speaking and writing activities that build concept and language attainment.

VOCABULARY

y-intercept, x-intercept, domain, range, exponential function, asymptote, base

MATERIALS

- Graphing calculator

STRATEGIES FOR SUCCESS

As students progress through the transformations on exponential functions, continue to ask them how the changes in *a*, *k*, *c*, and *d* relate to other functions. Emphasize that the parameters generate the same transformation, regardless of the function type. Use tables to show students numerically why the curves in the graph are changing.

1-6. *See page 228-A.*

9. *See page 228-A.*

INVESTIGATING *a*

- $Y1 = 2^x$
- $Y2 = 2 \cdot 2^x$
- $Y3 = 4 \cdot 2^x$
- $Y4 = 0.5 \cdot 2^x$
- $Y5 = 0.25 \cdot 2^x$

1. What transformation does *a* appear to create on the graph of $f(x) = 2^x$ when $a > 0$?
See margin.

- $Y1 = 2^x$
- $Y2 = -(2)^x$
- $Y3 = 0.5^x$
- $Y4 = -(0.5)^x$

2. What transformation does *a* appear to create on the graph of $f(x) = 2^x$ when $a < 0$?
See margin.

3. How do changes in *a* affect the domain and range of the exponential parent function, $f(x) = 2^x$?
See margin.

INVESTIGATING *k*

- $Y1 = 2^x$
- $Y2 = 2^{2x}$
- $Y3 = 2^{3x}$
- $Y4 = 2^{0.5x}$
- $Y5 = 2^{0.25x}$

4. What transformation does *k* appear to create on the graph of $f(x) = 2^x$ when $k > 0$?
See margin.

- $Y1 = 2^x$
- $Y2 = 2^{(-x)}$
- $Y3 = 2^{0.5x}$
- $Y4 = 2^{(-0.5x)}$

5. What transformation does *k* appear to create on the graph of $f(x) = 2^x$ when $k < 0$?
See margin.

6. How do changes in *k* affect the domain and range of the exponential parent function, $f(x) = 2^x$?
See margin.

INVESTIGATING *c* AND *d*

7. You have investigated the parameters *c* and *d* with several other function types. Make a conjecture about how these parameters will affect the graphs of exponential functions.
Possible response: The parameter c will horizontally translate the graph $\left|\frac{c}{b}\right|$ units and the parameter d will vertically translate the graph d units.

8. Graph the functions below to test your hypothesis. You may wish to use the *y*-intercept of the parent function to help look for patterns in the graph.

- $Y1 = 2^x$
- $Y2 = 2^{x-3}$
- $Y3 = 2^{x-5}$
- $Y4 = 2^{x+2}$
- $Y5 = 2^{x+4}$

- $Y1 = 2^x$
- $Y2 = 2^x + 2$
- $Y3 = 2^x + 5$
- $Y4 = 2^x - 3$
- $Y5 = 2^x - 6$

9. How does your conjecture compare the results of the graphs?
See margin.

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10. How do the values of c and d affect the domain and range of the exponential parent function, $f(x) = 2^x$?
See margin.

Use your investigations to answer the following questions.

11. Graph the following functions and use the graphs and table of values to explain how the y-intercept of each function compares to the values of c.
(Hint: Use the trace feature of your graphing technology to verify the y-coordinate when $x = 0$.)

- $Y1 = 2^x$
- $Y2 = 2^{x-2}$
- $Y3 = 2^{x-4}$
- $Y4 = 2^{x+2}$

See margin.

A negative exponent indicates that you should take the reciprocal of the base. For example, 2^{-3} is equivalent to $\left(\frac{1}{2}\right)^3 = \frac{1}{2^3}$.

12. Graph the following functions and use the graphs and table of values to explain how the y-intercept of each function compares to the values of both c and d. *(Hint: Use the trace feature of your graphing technology to verify the y-coordinate when $x = 0$.)*

- $Y1 = 2^{x-2}$
- $Y2 = 2^{x-2} + 1$
- $Y3 = 2^{x-2} + 3$
- $Y4 = 2^{x-2} - 3$

See margin.

13. Graph the following functions and use the graphs and table of values to explain how the y-intercept of each function compares to the values of a and k.

- $Y1 = 2^{(x-1)}$
- $Y2 = 2^{(0.5x-1)}$
- $Y3 = 3 \cdot 2^{(0.5x-1)}$
- $Y4 = 4 \cdot 2^{(0.5x-1)}$

See margin.

14. How could you write the coordinates of the y-intercept using the parameters a, k, c, and d for any exponential function, $f(x) = a(b)^{kx-c} + d$, with a base, b?
The y-intercept is the point $(0, \frac{a}{b^c} + d)$

REFLECT

- **How do vertical transformations (dilations, reflections, translations) affect the location of the horizontal asymptote of an exponential function?**
  See margin.
- **How does the asymptote relate to the minimum or maximum function value?**
  See margin.

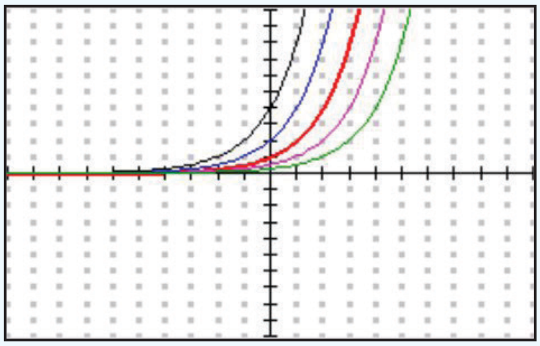
10–13. *See page 229-A.*

REFLECT ANSWERS:

A reflection moves the horizontal asymptote an equivalent distance across the x-axis, and a translation moves the horizontal asymptote vertically by d units. A reflection does not affect the horizontal asymptote.

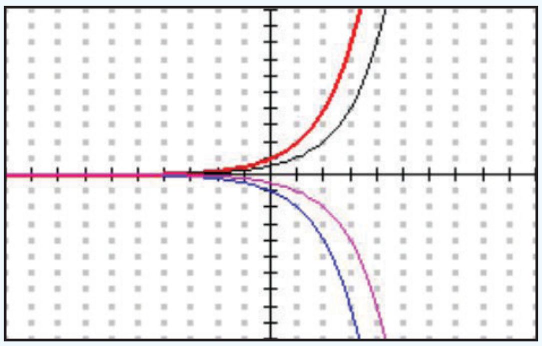
The asymptote represents a limit to the function values. For exponential functions where $a < 0$, the asymptote represents a number just above the maximum value of the function. For $a > 0$, the asymptote represents a number just below the minimum value of the function.

1. *The parameter a appears to create a vertical dilation. When |a| > 1, the dilation is a vertical stretch. When 0 < |a| < 1, the dilation is a vertical compression.*



X	Y1	Y2	Y3	Y4	Y5
0	1	2	4	.5	.25
1	2	4	8	1	.5
2	4	8	16	2	1
3	8	16	32	4	2
4	16	32	64	8	4
5	32	64	128	16	8
6	64	128	256	32	16
7	128	256	512	64	32
8	256	512	1024	128	64
9	512	1024	2048	256	128
10	1024	2048	4096	512	256

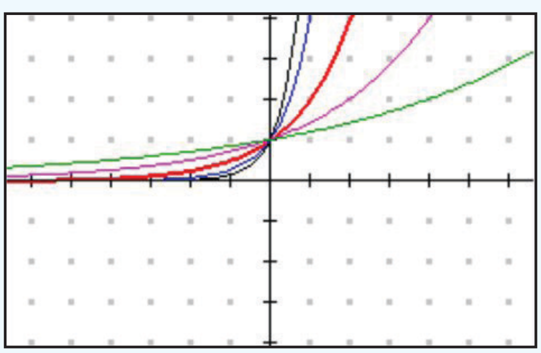
2. *The parameter a appears to create a vertical reflection across the x-axis when a < 0.*



X	Y1	Y2	Y3	Y4
0	1	-1	.5	-.5
1	2	-2	1	-1
2	4	-4	2	-2
3	8	-8	4	-4
4	16	-16	8	-8
5	32	-32	16	-16
6	64	-64	32	-32
7	128	-128	64	-64
8	256	-256	128	-128
9	512	-512	256	-256
10	1024	-1024	512	-512

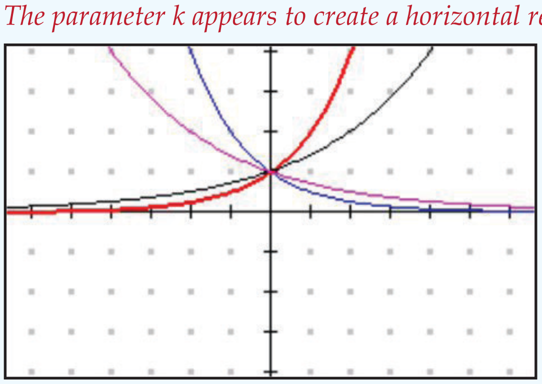
3. *The domain of the parent function is all real numbers and the range of the parent function is y > 0. When a is positive, the domain and range are not affected by a. However, when a is negative, the range becomes the opposite; e.g., if the range of $f(x) = a \cdot 2^x$ is y > 0, the range of $g(x) = -a \cdot 2^x$ is y < 0.*

4. *The parameter k appears to create a horizontal dilation. When |k| > 1, the dilation is a horizontal compression toward the y-axis by a factor of $\frac{1}{|k|}$. When 0 < |k| < 1, the dilation is a horizontal stretch away from the y-axis by a factor of $\frac{1}{|k|}$.*



X	Y1	Y2	Y3	Y4	Y5
0	1	1	1	1	1
1	2	4	8	1.4142	1.1892
2	4	16	64	2	1.4142
3	8	64	512	2.8284	1.6818
4	16	256	4096	4	2
5	32	1024	32768	5.6569	2.3784
6	64	4096	262144	8	2.8284
7	128	16384	2.1E6	11.314	3.3636
8	256	65536	1.68E7	16	4
9	512	262144	1.34E8	22.627	4.7568
10	1024	1.05E6	1.07E9	32	5.6569

5. *The parameter k appears to create a horizontal reflection across the y-axis when k < 0.*



X	Y1	Y2	Y3	Y4
0	1	1	1	1
1	2	.5	1.4142	.70711
2	4	.25	2	.5
3	8	.125	2.8284	.35355
4	16	.0625	4	.25
5	32	.03125	5.6569	.17678
6	64	.01563	8	.125
7	128	.00781	11.314	.08839
8	256	.00391	16	.0625
9	512	.00195	22.627	.04419
10	1024	9.8E-4	32	.03125

6. *The domain and range of the parent function are not affected by changes in k. The domain remains all real numbers and the range remains y > 0.*

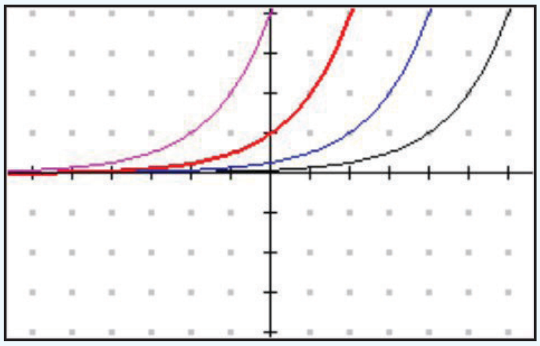
9. *Possible response: The values of b and c translates the graph horizontally by $\frac{c}{b}$ units. When c < 0, the graph is translated to the left $\left|\frac{c}{b}\right|$ units and when c > 0, the graph is translated to the right $\left|\frac{c}{b}\right|$ units.*

The value of d translates the graph vertically by d units. When d < 0, the graph is translated down |d| units and when d > 0, the graph is translated up |d| units.

10. *The value of c does not affect the domain or range of the parent function.*

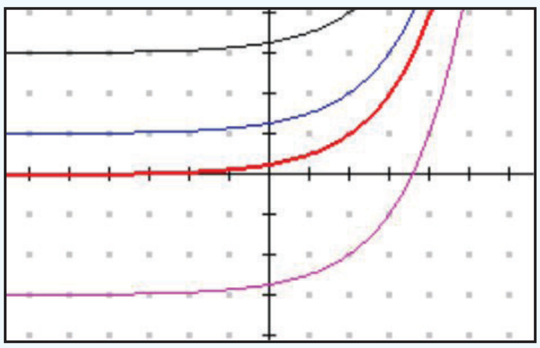
The value of d vertically shifts the horizontal asymptote of the parent function (for $f(x) = 2^x$, the horizontal asymptote is $y = 0$, or the x-axis) d units, which vertically shifts the range to $y > d$ if $a > 0$ or $y < d$ if $a < 0$.

11. *The value of c places the y-intercept 2^{-c} units above the horizontal asymptote when $a > 0$ and below the horizontal asymptote when $a < 0$.*



X	Y1	Y2	Y3	Y4
0	1	.25	.0625	4
1	2	.5	.125	8
2	4	1	.25	16
3	8	2	.5	32
4	16	4	1	64
5	32	8	2	128
6	64	16	4	256
7	128	32	8	512
8	256	64	16	1024
9	512	128	32	2048
10	1024	256	64	4096

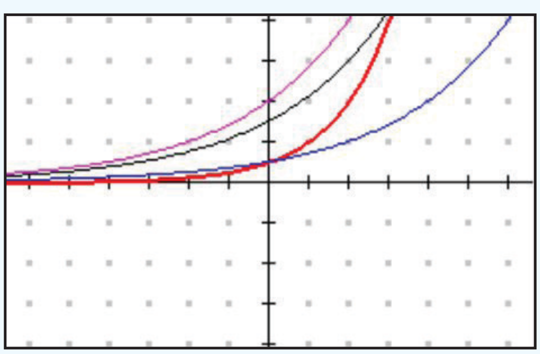
12. *The value of c places the y-intercept 2^{-c} units above the horizontal asymptote and the value of d places the y-intercept d units above the horizontal asymptote. Together, the y-intercept is moved a combination of $(2^{-c} + d)$ units.*



X	Y1	Y2	Y3	Y4
0	.25	1.25	3.25	-2.75
1	.5	1.5	3.5	-2.5
2	1	2	4	-2
3	2	3	5	-1
4	4	5	7	1
5	8	9	11	5
6	16	17	19	13
7	32	33	35	29
8	64	65	67	61
9	128	129	131	125
10	256	257	259	253

x	$Y1 = 2^{x-2}$	$Y2 = 2^{x-2} + 1$	$Y3 = 2^{x-2} + 3$	$Y4 = 2^{x-2} - 3$
0	$2^{0-2} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$2^{0-2} + 1 = 2^{-2} + 1 = \frac{1}{2^2} + 1 = \frac{1}{4} + 1 = 1.25$	$2^{0-2} + 3 = 2^{-2} + 3 = \frac{1}{2^2} + 3 = \frac{1}{4} + 3 = 3.25$	$2^{0-2} - 3 = 2^{-2} - 3 = \frac{1}{2^2} - 3 = \frac{1}{4} - 3 = -2.75$

13. *The value of k does not affect the y-intercept and the value of a multiplies the vertical movement generated by c.*



X	Y1	Y2	Y3	Y4
0	.5	.5	1.5	2
1	1	.70711	2.1213	2.8284
2	2	1	3	4
3	4	1.4142	4.2426	5.6569
4	8	2	6	8
5	16	2.8284	8.4853	11.314
6	32	4	12	16
7	64	5.6569	16.971	22.627
8	128	8	24	32
9	256	11.314	33.941	45.255
10	512	16	48	64

x	$Y1 = 2^{x-1}$	$Y2 = 2^{(0.5x-1)}$	$Y3 = 3 \cdot 2^{(0.5x-1)}$	$Y4 = 4 \cdot 2^{(0.5x-1)}$
0	$2^{0-1} = 2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$2^{0.5(0)-1} = 2^{0-1} = 2^{-1} = \frac{1}{2^1} = \frac{1}{2} = 0.5$	$3 \cdot 2^{0.5(0)-1} = 3 \cdot 2^{0-1} = 3 \cdot 2^{-1} = \frac{3}{2^1} = \frac{3}{2} = 1.5$	$4 \cdot 2^{0.5(0)-1} = 4 \cdot 2^{0-1} = 4 \cdot 2^{-1} = \frac{4}{2^1} = \frac{4}{2} = 2$

REFLECT ANSWER:

The horizontal asymptote does not affect the domain, but does place a boundary on the range. When $a > 0$, the range includes all the y-values above the horizontal asymptote and when $a < 0$, the range includes all the y-values below the horizontal asymptote.

ADDITIONAL EXAMPLES

Determine whether the functions below represent exponential growth or exponential decay.

1. $y = 3.5^x$
Exponential Growth

2. $y = \left(\frac{2}{3}\right)^x$
Exponential Decay

3. $y = \left(\frac{5}{4}\right)^x$
Exponential Growth

4. $y = 0.36^x$
Exponential Decay

- **How does the asymptote affect the domain and range of rational functions?**
  See margin.

EXPLAIN

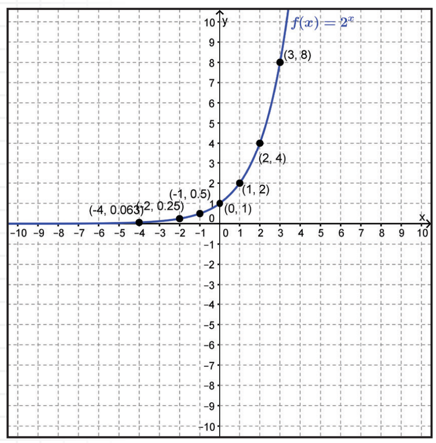
An exponential function is a function that uses a constant multiplier, or base, to show either growth or decay. Exponential growth functions have a multiplier, or base, that is greater than 1. Exponential decay functions have a multiplier, or base, that is between 0 and 1.

Watch Explain and You Try It Videos

or click here

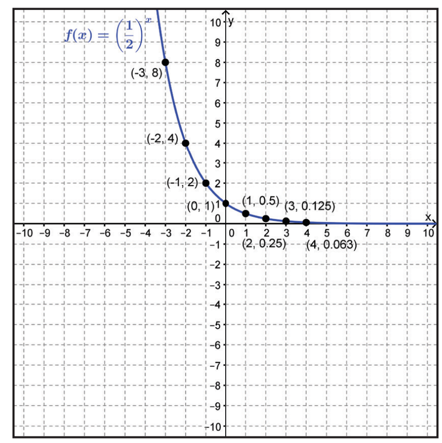
Both exponential growth and decay functions can be written using the parameters a, k, c, and d in the general form $f(x) = a(b)^{kx-c} + d$.

$f(x) = 2^x$



x	-4	-2	-1	0	1	2	3
f(x)	$\frac{1}{16}$	$\frac{1}{4}$	0.5	1	2	4	8

$f(x) = \left(\frac{1}{2}\right)^x$



x	-3	-2	-1	0	1	2	3	4
f(x)	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

CHANGES IN *a*

The parameter a influences the vertical dilation of the graph.

- If $|a| > 1$, then the range values (y-coordinates) of the original function are multiplied by a factor of $|a|$ in order to vertically stretch the graph.
- If $0 < |a| < 1$, then the range values (y-coordinates) of the original function are multiplied by a factor of $|a|$ in order to vertically compress the graph.
- If $a < 0$, then the graph will be reflected across the x-axis.

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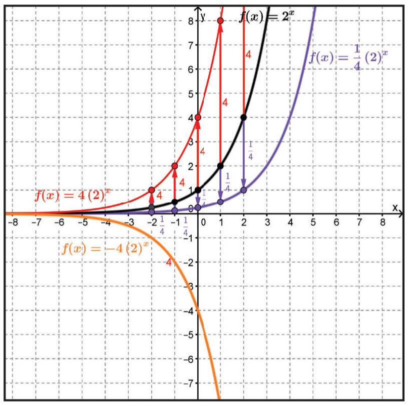


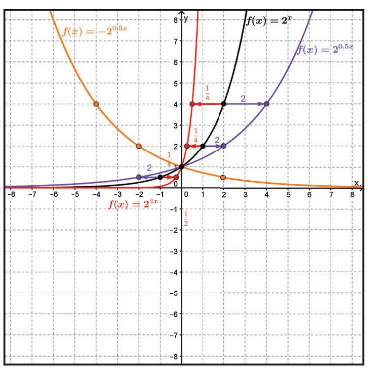
x	$f(x) = 2^x$	$f(x) = 4 \cdot 2^x$	$f(x) = \frac{1}{4}(2)^x$	$f(x) = -4 \cdot 2^x$
-2	$\frac{1}{4}$	1	$\frac{1}{16}$	-1
-1	$\frac{1}{2}$	2	$\frac{1}{8}$	-2
0	1	4	$\frac{1}{4}$	-4
1	2	8	$\frac{1}{2}$	-8
2	4	16	1	-16

× 4   × $\frac{1}{4}$ × (-1)

CHANGES IN k

The parameter k influences the horizontal stretch or compression of the graph.

- If $|k| > 1$, then the domain values (x-coordinates) of the original function are multiplied by a factor of $\frac{1}{|k|}$, which will be a multiplier that is less than 1, in order to horizontally compress the graph.
- If $0 < |k| < 1$, then the domain values (x-coordinates) of the original function are multiplied by a factor of $\frac{1}{|k|}$, which will be a multiplier that is greater than 1, in order to horizontally stretch the graph.



x	$f(x) = 2^x$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4

x	$f(x) = 2^{4x}$
$-\frac{1}{2}$	$\frac{1}{4}$
$-\frac{1}{4}$	$\frac{1}{2}$
0	1
$\frac{1}{4}$	2
$\frac{1}{2}$	4

Domain (x) values are multiplied by $\frac{1}{4}$ in order to generate the same range (y) value. This multiplication results in a horizontal compression of the graph.

x	$f(x) = 2^x$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4

x	$f(x) = 2^{0.5x}$
-4	$\frac{1}{4}$
-2	$\frac{1}{2}$
0	1
2	2
4	4

Domain (x) values are multiplied by 2 in order to generate the same range (y) value. This multiplication results in a horizontal stretch of the graph.

If $k < 0$, then all of the x-values will change signs and the graph will be reflected across the y-axis.

Domain (x) values are multiplied by −1, which changes the sign but not the magnitude of each x-value. This change results in a horizontal reflection of the graph across the y-axis.

x	$f(x) = 2^{0.5x}$
-4	$\frac{1}{4}$
-2	$\frac{1}{2}$
0	1
2	2
4	4

x	$f(x) = 2^{(-0.5x)}$
4	$\frac{1}{4}$
2	$\frac{1}{2}$
0	1
-2	2
-4	4

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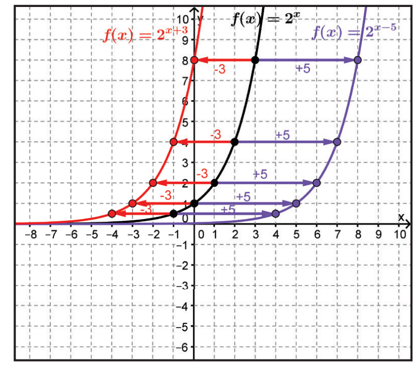
CHANGES IN c

The parameters k and c influence generates a horizontal translation of the graph.

- If $c > 0$, then the graph will translate $\left|\frac{c}{k}\right|$ units to the right.
- If $c < 0$, then the graph will translate $\left|\frac{c}{k}\right|$ units to the left.



x	$f(x) = 2^x$	x	$f(x) = 2^{x-5}$
-1	$\frac{1}{2}$	4	$\frac{1}{2}$
0	1	5	1
1	2	6	2
2	4	7	4
3	8	8	8

Domain (x) values are increased by 5 in order to generate the same range (y) value. This addition results in a horizontal translation of the graph to the right.

x	$f(x) = 2^x$	x	$f(x) = 2^{x+3}$
-1	$\frac{1}{2}$	-4	$\frac{1}{2}$
0	1	-3	1
1	2	-2	2
2	4	-1	4
3	8	0	8

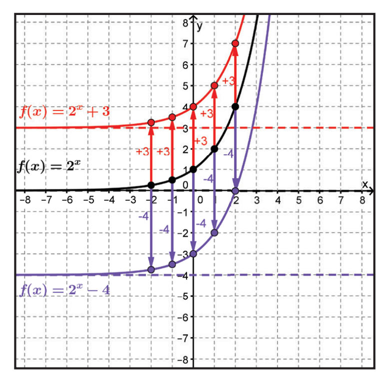
Domain (x) values are decreased by 3 in order to generate the same range (y) value. This subtraction (addition with a negative number) results in a horizontal translation of the graph to the left.

CHANGES IN d

The parameter d generates a vertical translation of the graph.

- If $d > 0$, then the graph will translate $|d|$ units up.
- If $d < 0$, then the graph will translate $|d|$ units down.

Notice that the horizontal asymptote, $y = 0$, of the parent function is also translated vertically with the graph. This shift will impact the range restriction for an exponential function.



x	$f(x) = 2^x$	$f(x) = 2^x - 4$	$f(x) = 2^x + 3$
-2	$\frac{1}{4}$	$-3\frac{3}{4}$	$3\frac{1}{4}$
-1	$\frac{1}{2}$	$-3\frac{1}{2}$	$3\frac{1}{2}$
0	1	-3	4
1	2	-2	5
2	4	0	7

−4 +3

Each of these parameters also has an effect on key attributes of an exponential function.

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ASYMPTOTE

Each exponential function has a horizontal asymptote. For the parent exponential growth and decay functions, the horizontal asymptote is $y = 0$, or the x-axis.

The horizontal asymptote is governed by vertical parameter changes. A vertical translation moves the asymptote d units, but a horizontal dilation does not move the asymptote.

Horizontal asymptote: $y = d$

DOMAIN AND RANGE

Exponential functions do not have any domain restrictions. That is, all real numbers could be used for the exponent in the exponential function. The domain of any exponential function is all real numbers, $\{x \mid x \in \mathbb{R}\}$

In terms of the range, let's look more closely at $f(x) = a(b)^{kx-c} + d$. The base of the exponential function, b, is a positive number. Raising a positive number to any power yields another positive number, so the function values for an exponential function, $f(x) = b^{kx-c}$, must all be greater than 0. Hence, exponential functions that are not vertically reflected or translated have an asymptote at $y = 0$, or the x-axis.

The parameters a or d could reflect or translate the exponential function beyond the line $y = 0$, moving the horizontal asymptote along with the rest of the function. The equation of the asymptote, $y = d$, is the boundary of the range. If $a > 0$, then the range is $y > d$. If $a < 0$, then the graph is reflected across the x-axis and the range becomes $y < d$.

Domain: $\{x \mid x \in \mathbb{R}\}$
Range: If $a > 0$, $\{y \mid y \in \mathbb{R}, y > d\}$; If $a < 0$, $\{y \mid y \in \mathbb{R}, y < d\}$

y-INTERCEPT

The y-intercept of any function is the point $(0, y)$, or the point where the graph intersects the y-axis. For an exponential function, you can calculate the y-intercept by substituting $x = 0$ into the general form, $y = a(b)^{kx-c} + d$.

$$y = a(b)^{kx-c} + d$$
$$y = a(b)^{k(0)-c} + d$$
$$y = a(b)^{-c} + d$$
$$y = \frac{a}{b^c} + d$$

The coordinates of the y-intercept are $(0, \frac{a}{b^c} + d)$.

The parameters a, c, and d each influence the y-intercept of an exponential function. The parameter d vertically translates the entire graph, including the y-intercept, and the parameter c horizontally translates the entire graph, including the y-intercept. The parameter a vertically dilates or reflects the entire graph, including the y-intercept. Each of these transformations affects where the graph intersects the y-axis.

STRATEGIES FOR SUCCESS

Because b stands for base in rational functions, students may become easily confused between b and k when finding x- and y-intercepts. Have students take special note of the b in these two formulas denoting base, not x's coefficient.

KEY ATTRIBUTES OF EXPONENTIAL FUNCTIONS

An exponential function has several important key attributes:

- The domain of an exponential function is all real numbers. $\{x \mid x \in \mathbb{R}\}$
- If $a > 0$, the range of an exponential function is a set of real numbers greater than the horizontal asymptote. $\{y \mid y > d\}$
- If $a < 0$, the range of an exponential function is a set of real numbers less than the horizontal asymptote. $\{y \mid y < d\}$
- The horizontal asymptote of an exponential function is $y = d$.
- An exponential function may have one x-intercept or none.
- An exponential function has one y-intercept at $(0, \frac{a}{b^c} + d)$.
- The horizontal asymptote is a boundary for the function. If $a > 0$, then the minimum function value will not decrease to or go below the asymptote. If $a < 0$, then the maximum function value will not increase to or go above the asymptote.

EXAMPLE 1

What transformations of the exponential parent function, $f(x) = (10)^x$, will result in the graph of the exponential function $g(x) = -3(10)^{2x-1} + 5$?

STEP 1 **Determine the values of the parameters a, k, c and d of $g(x)$ and the value of b, the base of $g(x)$.**

$a = -3$, $b = 10$, $k = 2$, $c = 1$, and $d = 5$.

STEP 2 **Use the values of the parameters to describe the transformations of the exponential parent function $f(x)$ that are necessary to produce $g(x)$.**

$a = -3$, so $|a| > 0$. The dependent variable values (y-coordinates) of the exponential parent function are multiplied by a factor of 3 in order to vertically stretch the graph of the function. Additionally, since a is negative, the graph is reflected across the x-axis.

$b = 10$, which is a multiplier or base greater than 1, so this is an exponential growth function.

$k = 2$, so $k > 1$. The independent variable values (x-coordinates) of the original function are multiplied by a factor of $\frac{1}{2}$, which will horizontally compress the graph.

ADDITIONAL EXAMPLES

What transformations of the exponential parent function, $p(x) = \left(\frac{1}{2}\right)^x$ will result in the graph of the exponential functions below?

1. $q(x) = \frac{2}{3}\left(\frac{1}{2}\right)^{(-x+3)} - 4$

The graph of q(x) is produced by transforming the exponential decay parent function by vertically compressing its graph by a factor of $\frac{2}{3}$, reflecting its graph across the y-axis, and translating its graph 3 units to the right and 4 units down.

2. $r(x) = -5\left(\frac{1}{2}\right)^{(3x-8)} + 6.5$

The graph of r(x) is produced by transforming the exponential decay parent function by vertically stretching its graph by a factor of 5, horizontally compressing its graph by a factor of $\frac{1}{3}$, reflecting its graph across the x-axis, and translating its graph $\frac{8}{3}$ units to the right and 6.5 units up.

3. $v(x) = \left(\frac{1}{2}\right)^{(0.25x+2)}$

The graph of v(x) is produced by transforming the exponential decay parent function by horizontally stretching its graph by a factor of 4 and translating its graph 8 units to the left.

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The parameter c generates a horizontal translation of the graph. Since $c = 1$, $c > 0$, and the graph will translate $\left|\frac{1}{2}\right| = \frac{1}{2}$ unit to the right.

$d = 5$, so $d > 0$. The graph of the exponential parent function will translate 5 units upward.

The graph of g(x) is produced by transforming the exponential growth parent function f(x) by vertically stretching its graph by a factor of three, horizontally compressing its graph by a factor of one half, reflecting its graph over the x-axis, and translating its graph one-half unit to the right and five units upward.

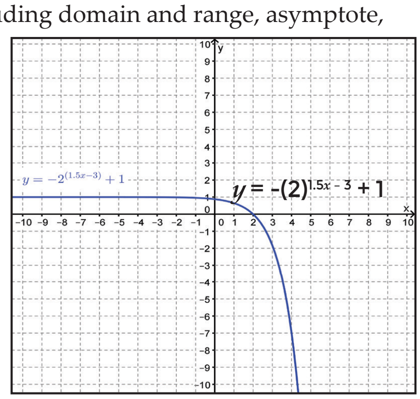
YOU TRY IT! #1

What transformations of the exponential parent function, $y = (3)^x$, will result in the graph of the exponential function $y = \frac{5}{6}(3)^{x+1}$?

See margin.

EXAMPLE 2

Identify the key attributes of $y = -(2)^{1.5x-3} + 1$, including domain and range, asymptote, x-intercept, and y-intercept. Write the domain and range in set builder notation.



STEP 1 **Determine the domain, range and asymptote of $y = -(2)^{1.5x-3} + 1$.**

From the graph, we can see the domain contains all real values of x. This is true for exponential functions. Written in set builder notation, the domain is $\{x \mid x \in \mathbb{R}\}$.

From the equation, the value of $d = 1$ indicates that the graph of this exponential function would have a horizontal asymptote at $y = 1$. Also, since a is negative, the graph is vertically reflected over the x-axis. This is confirmed by the provided graph. Since $d = 1$, the range of the function contains all real values of $y < 1$. This is written in set builder notation as $\{y \mid y < 1\}$.

YOU TRY IT! #1 ANSWER:

The graph of $y = \frac{5}{6}(3)^{x+1}$ is produced by transforming the exponential growth parent function by vertically compressing its graph by a factor of $\frac{5}{6}$ and translating its graph one unit to the left. Since $k = 1$, there is no horizontal dilation and no reflection over the y-axis. Since $d = 0$, there is no vertical translation. And since a is positive, there is no reflection over the x-axis.

ADDITIONAL EXAMPLE

Identify the key attributes of $y = \frac{1}{3}\left(\frac{1}{4}\right)^{(5x+10)} - 7$, including domain and range, asymptotes, x-intercept, and y-intercept. Write the domain and range in set builder notation.

The domain of the exponential decay function is $\{x \mid x \in \mathbb{R}$,. The range is $\{y \mid y > -7\}$. The horizontal asymptote is $y = -7$. The x-intercept occurs between $x = -3$ and $x = -2$. The y-intercept is about $(0, -7)$.

INSTRUCTIONAL HINTS

Students who are struggling to find the y-intercept are likely confusing b and k. Have students write the standard form of a rational function at the top of their paper as they work. Make sure students identify a, b, c, d, and k prior to using their values in a formulas.

STEP 2 **Determine if the function has an x-intercept.**

The x-intercept of an exponential function is located where $y = 0$. According to the graph, the x-intercept occurs at (2, 0).

STEP 3 **Determine if the function has a y-intercept.**

The y-intercept of an exponential function is located where $x = 0$, $y = \frac{a}{b^c} + d$.

$y = -(2)^{1.5x - 3} + 1$

$y = -(2)^{1.5(0) - 3} + 1$

$y = -(2)^{-3} + 1$

$y = -\frac{1}{8} + 1 = \frac{7}{8}$

So the y-intercept of $y = -(2)^{1.5x - 3} + 1$ is $(0, \frac{7}{8})$. The graph confirms this.

The domain of the exponential function $y = -(2)^{1.5x - 3} + 1$ is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y < 1\}$. The horizontal asymptote is $y = 1$. The x-intercept occurs at (2, 0). The y-intercept is $(0, \frac{7}{8})$.

YOU TRY IT! #2

Identify the key attributes of $y = -3\left(\frac{1}{2}\right)^x + 5$, including domain and range, asymptote, x-intercept, and y-intercept. Write the domain and range in set builder notation.

See margin.

x	y
-5	-91
-4	-43
-3	-19
-2	-7
-1	-1
0	2
1	3.5
2	4.25
3	4.625
4	4.8125
5	4.9063

YOU TRY IT! #2 ANSWER:

The domain of the exponential function $y = -3\left(\frac{1}{2}\right)^x + 5$ is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y < 5\}$. The horizontal asymptote is $y = 5$. The x-intercept occurs between $x = -1$ and $x = 0$. The y-intercept is (0, 2).

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EXAMPLE 3

Identify and compare asymptotes of $f(x) = 5^{4x-1}$ and $g(x) = \frac{5}{4x-1}$.

STEP 1 **Determine the asymptotes of the exponential function $f(x) = 5^{4x-1}$.**

Exponential functions have a horizontal asymptote at $y = d$. In the function $f(x) = 5^{4x-1}$, $d = 0$, and so the horizontal asymptote is $y = 0$.

STEP 2 **Determine the asymptotes of the rational function $g(x) = \frac{5}{4x-1}$.**

Rational functions have both a horizontal and a vertical asymptote.

The horizontal asymptote occurs at $y = d$. In the function $(x) = \frac{5}{4x-1}$, $d = 0$, and so the horizontal asymptote is $y = 0$.

The vertical asymptote occurs at $x = \frac{c}{b}$. In the function $(x) = \frac{5}{4x-1}$, $c = 1$ and $b = 4$, and so the vertical asymptote is $x = \frac{1}{4}$.

The horizontal asymptote of the exponential function $f(x) = 5^{4x-1}$ is $y = 0$. The horizontal asymptote of the rational function $g(x) = \frac{5}{4x-1}$ is also $y = 0$. The vertical asymptote of the rational function $g(x) = \frac{5}{4x-1}$ is $x = \frac{1}{4}$.

YOU TRY IT! #3

Identify and compare asymptotes of the exponential function $f(x) = -0.25^x - 3.5$ and the rational function $g(x) = \frac{-0.25}{x} - 3.5$.

See margin.

EXAMPLE 4

Identify and compare the domains and ranges as well as any existing asymptotes of $f(x) = 2(x + 7) - 3$, $g(x) = 2^{x+7} - 3$, and $h(x) = \frac{1}{2x+7} - 3$. Write the domain and range of each function as inequalities and in set builder notation.

STEP 1 **Determine the domain and range of $f(x)$.**

ADDITIONAL EXAMPLE

Identify and compare asymptotes of $h(x) = 0.5^{(0.4x+8)} - 3$ and $j(x) = \frac{0.5}{0.4x+8} - 3$.

The horizontal asymptote of both the exponential decay function and rational function is $y = -3$. The exponential decay function does not have a vertical asymptote. The vertical asymptote of the rational function is $x = -20$.

YOU TRY IT! #3 ANSWER:

The horizontal asymptote of the exponential function $f(x) = -0.25^x - 3.5$ is $y = -3.5$. The horizontal asymptote of the rational function $g(x) = \frac{-0.25}{x} - 3.5$ is also $y = -3.5$. The vertical asymptote of the rational function $g(x) = \frac{-0.25}{x} - 3.5$ is $x = 0$.

Since $f(x) = 2(x + 7) - 3$ is a linear function, its domain contains all real numbers, $-\infty < x < \infty$. Written in set builder notation, $\{x \mid x \in \mathbb{R}\}$. Its range also contains all real numbers, $-\infty < f(x) < \infty$. This written in set builder notation is $\{f(x) \mid f(x) \in \mathbb{R}\}$. Linear functions have no asymptotes.

STEP 2 **Determine the domain and range of $g(x)$.**

Since $g(x) = 2^{x+7} - 3$ is an exponential function, its domain contains all real numbers, $-\infty < x < \infty$. Written in set builder notation, $\{x \mid x \in \mathbb{R}\}$.

The function's range also contains real numbers, bounded by a horizontal asymptote, where $g(x) = d$. In $g(x) = 2^{x+7} - 3$, $d = -3$. Since a is positive, the values of y lie above the horizontal asymptote. So the range is $-3 < g(x) < \infty$, which written in set builder notation is $\{g(x) \mid g(x) > -3\}$.

STEP 3 **Determine the domain and range of $h(x)$.**

Since $h(x) = \frac{1}{2x+7} - 3$ is a rational function, its domain contains all real numbers except for $x = \frac{c}{b}$. In this function, $c = -7$ and $b = 2$, so as an inequality, this is written as $-\infty < x < \frac{-7}{2}$ or $\frac{-7}{2} < x < \infty$. Written in set builder notation, this is $\{x \mid x \in \mathbb{R}, x \neq -\frac{7}{2}\}$. Related to this restriction to the range is the vertical asymptote, $x = \frac{-7}{2}$.

The range of the rational function contains all real numbers except for $y = d$. In $h(x) = \frac{1}{2x+7} - 3$, $d = -3$, which is related to the horizontal asymptote, $y = -3$. Since $a = 1$ and is greater than zero, the graph is not reflected over the x-axis. The range, written as an inequality, is $-\infty < g(x) < -3$ or $-3 < g(x) < \infty$. Written in set builder notation, this is $\{h(x) \mid h(x) \in \mathbb{R}, h(x) \neq -3\}$.

The domains of first two functions are the same, all real numbers, whereas h(x) is a rational function and its domain excludes the x-value related to its vertical asymptote, $x = \frac{-7}{2}$. The range of the exponential function g(x) does not contain all real numbers but is restricted to only those real numbers greater than - 3, whereas the linear function f(x) has no restrictions. The range of the rational function h(x) excludes the y-value related to its horizontal asymptote, y = – 3.

ADDITIONAL EXAMPLE

Identify and compare the domains and ranges of the given functions as well as any existing asymptotes.

$j(x) = -0.8(5x - 4)^2 + 7$

$k(x) = -0.8(5x - 4)^3 + 7$

$n(x) = -0.8^{(5x-4)} + 7$

Write the domain and range of each function as inequalities and in set builder notation.

The domains of j(x), k(x), and n(x) are the same, all real numbers which can be written as $\{x \mid x \in \mathbb{R}\}$.

The range of j(x) is all real numbers, $\{j(x) \mid j(x) \in \mathbb{R}\}$, but the range of k(x) and n(x) have restrictions. The range of k(x) is $\{k(x) \mid k(x) \in \mathbb{R}\}$. The range of n(x) is $\{n(x) \mid n(x) < 7\}$. n(x) has one horizontal asymptote at y = 7.

	NOTATION	$j(x) = -0.8(5x - 4)^2 + 7$	$k(x) = -0.8(5x - 4)^3 + 7$	$n(x) = -0.8^{(5x-4)} + 7$
DOMAIN	INEQUALITY	$-\infty < x < \infty$	$-\infty < x < \infty$	$-\infty < x < \infty$
	SET BUILDER	$\{x \mid x \in \mathbb{R}\}$	$\{x \mid x \in \mathbb{R}\}$	$\{x \mid x \in \mathbb{R}\}$
RANGE	INEQUALITY	$-\infty < j(x) < \infty$	$-\infty < k(x) \leq \infty$	$-\infty < n(x) < 7$
	SET BUILDER	$\{j(x) \mid j(x) \in \mathbb{R}\}$	$\{k(x) \mid k(x) \in \mathbb{R}\}$	$\{n(x) \mid n(x) < 7\}$
ASYMPTOTE	VERTICAL	NONE	NONE	NONE
	HORIZONTAL	NONE	NONE	$n(x) = 7$

		$f(x) = 2(x+7) - 3$	$g(x) = 2^{x+7} - 3$	$h(x) = \frac{1}{2x+7} - 3$
	NOTATION	$f(x) = 2(x+7) - 3$	$g(x) = 2^{x+7} - 3$	$h(x) = \frac{1}{2x+7} - 3$
DOMAIN	**INEQUALITY**	$-\infty < x < \infty$	$-\infty < x < \infty$	$-\infty < x < \frac{-7}{2}$ or $\frac{-7}{2} < x < \infty$
DOMAIN	**SET BUILDER**	$\{x \mid x \in \mathbb{R}\}$	$\{x \mid x \in \mathbb{R}\}$	$\{x \mid x \in \mathbb{R}, x \neq \frac{-7}{2}\}$
RANGE	**INEQUALITY**	$-\infty < f(x) < \infty$	$-3 < g(x) < \infty$	$-\infty < h(x) < -3$ or $-3 < h(x) < \infty$
RANGE	**SET BUILDER**	$\{f(x) \mid f(x) \in \mathbb{R}\}$	$\{g(x) \mid g(x) > -3\}$	$\{h(x) \mid h(x) \in \mathbb{R}, h(x) \neq -3\}$
ASYMPTOTE	**VERTICAL**	NONE	NONE	$x = \frac{-7}{2}$
ASYMPTOTE	**HORIZONTAL**	NONE	$g(x) = -3$	$h(x) = -3$

YOU TRY IT! #4

Identify and compare the domains and ranges as well as any existing asymptotes of $f(x) = 3(2x+1) + 15$, $g(x) = 3^{2x+1} + 15$, and $h(x) = \frac{3}{2x+1} + 15$. Write the domain and range of each function as inequalities and in set builder notation.

See margin.

PRACTICE/HOMEWORK

For questions 1 – 4 determine whether the following exponential functions represent exponential growth or exponential decay.

1. $h(x) = 1.5^x$ **Exponential growth**
2. $g(x) = \left(\frac{2}{3}\right)^x$ **Exponential decay**
3. $h(x) = 9^x$ **Exponential growth**
4. $g(x) = \left(\frac{7}{3}\right)^x$ **Exponential growth**

For questions 5 - 8, describe what transformations of the parent exponential function $f(x) = 2^x$, will result in the graph of the given function.

5. $g(x) = \frac{3}{4}(2)^x + 1$ **See margin.**
6. $h(x) = (2)^{3x} - 6$ **See margin.**
7. $g(x) = (2)^{0.5x+3}$ **See margin.**
8. $h(x) = -(2)^{2x} + 5$ **See margin.**

5. *Transform the exponential function, $f(x) = 2^x$, by vertically compressing its graph by a factor of $\frac{3}{4}$, and translating its graph 1 unit up.*

6. *Transform the exponential function, $f(x) = 2^x$, by horizontally compressing its graph by a factor of $\frac{1}{3}$, and translating its graph 6 units down.*

7. *Transform the exponential function, $f(x) = 2^x$, by horizontally stretching its graph by a factor of 2, and translating its graph 6 units to the left.*

8. *Transform the exponential function, $f(x) = 2^x$, by reflecting its graph across the x-axis, horizontally compressing it by a factor of $\frac{1}{2}$, and translating its graph 5 units up.*

YOU TRY IT! #4 ANSWER:

The domains of first two functions are the same, all real numbers, whereas $h(x)$ is a rational function and its domain excludes the x-value related to its vertical asymptote, $x = -\frac{1}{2}$. The range of the exponential function $g(x)$ does not contain all real numbers but is restricted to only those real numbers greater than 15, whereas the linear function $f(x)$ has no restrictions. The range of the rational function $h(x)$ excludes the y-value related to its horizontal asymptote, $y = 15$.

	NOTATION	$f(x) = 3(2x+1) + 15$	$g(x) = 3^{2x+1} + 15$	$h(x) = \frac{3}{2x+1} + 15$
DOMAIN	**INEQUALITY**	$-\infty < x < \infty$	$-\infty < x < \infty$	$-\infty < x < -\frac{1}{2}$ or $-\frac{1}{2} < x < \infty$
DOMAIN	**SET BUILDER**	$\{x \mid x \in \mathbb{R}\}$	$\{x \mid x \in \mathbb{R}\}$	$\{x \mid x \in \mathbb{R}, x \neq -\frac{1}{2}\}$
RANGE	**INEQUALITY**	$-\infty < f(x) < \infty$	$15 < g(x) < \infty$	$-\infty < h(x) < 15$ or $15 < h(x) < \infty$
RANGE	**SET BUILDER**	$\{f(x) \mid f(x) \in \mathbb{R}\}$	$\{g(x) \mid g(x) > 15\}$	$\{h(x) \mid h(x) \in \mathbb{R}, h(x) \neq 15\}$
ASYMPTOTE	**VERTICAL**	NONE	NONE	$x = -\frac{1}{2}$
ASYMPTOTE	**HORIZONTAL**	NONE	$g(x) = 15$	$h(x) = 15$

For questions 9 - 12, describe what transformations of the parent exponential function, $f(x)=\left(\frac{1}{2}\right)^x$ will result in the graph of the given function.

9. $h(x)=9\left(\frac{1}{2}\right)^{x-5}$ **See margin.**

10. $g(x)=-\left(\frac{1}{2}\right)^x+3$ **See margin.**

11. $h(x)=-3\left(\frac{1}{2}\right)^x-7$ **See margin.**

12. $g(x)=\left(\frac{1}{2}\right)^{4x}+10$ **See margin.**

For questions 13 – 16, match the exponential function to its graph or table on the right.

13. $g(x)=5\left(\frac{1}{2}\right)^x-4$
C

14. $h(x)=-(2)^x-5$
B

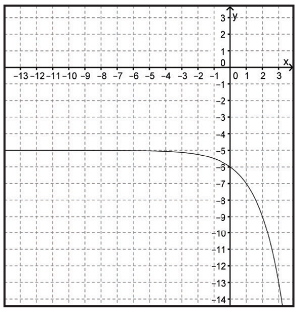
15. $g(x)=-4(2)^x$
A

16. $h(x)=6\left(\frac{1}{2}\right)^x+1$
D

A.

x	y
-3	-0.5
-2	-1
-1	-2
0	-4
1	-8
2	-16

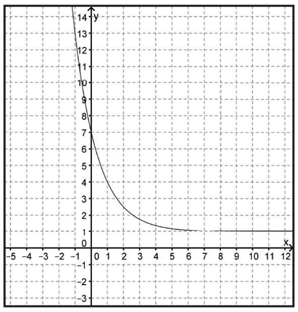
B.



C.

x	y
-3	36
-2	16
-1	6
0	1
1	-1.5
2	-2.75

D.



For each exponential function in questions 17 – 20, determine the domain and range, horizontal asymptote, x-intercept, and y-intercept. Write the domain and range in set builder notation.

17. $g(x)=15\left(\frac{1}{2}\right)^x-4$ **See margin.**

18. $h(x)=-(3)^{x+1}-5$ **See margin.**

19. $g(x)=6(1.5)^{2x}$ **See margin.**

20. $g(x)=-\left(\frac{3}{4}\right)^{3x-4}+7$ **See margin.**

For questions 21 – 23, determine and compare the asymptotes of the given sets of functions.

21. $h(x)=(3)^{x+1}-7$ and $g(x)=\frac{3}{x+1}-7$
The horizontal asymptote of both functions is y = -7.
The vertical asymptote of g(x) is x = -1.

22. $h(x)=\frac{-0.25}{x}+1$ and $g(x)=-(0.25)^x+1$
The horizontal asymptote of both functions is y = 1.
The vertical asymptote of h(x) is x = 0.

23. $h(x)=2^{3x+5}$ and $g(x)=\frac{2}{3x+5}$
The horizontal asymptote of both functions is y = 0.
The vertical asymptote of g(x) is $x=-\frac{5}{3}$.

9. *Transform the exponential function, $f(x)=\left(\frac{1}{2}\right)^x$, by vertically stretching its graph by a factor of 9, and translating its graph 5 units to the right.*

10. *Transform the exponential function, $f(x)=\left(\frac{1}{2}\right)^x$, by reflecting its graph across the x-axis, and translating its graph 3 units up.*

11. *Transform the exponential function, $f(x)=\left(\frac{1}{2}\right)^x$, by reflecting its graph across the x-axis, vertically stretching it by a factor of 3, and translating its graph 7 units down.*

12. *Transform the exponential function, $f(x)=\left(\frac{1}{2}\right)^x$, by horizontally compressing its graph by a factor of $\frac{1}{4}$, and translating its graph 10 units up.*

17. *The domain is $\{x \mid x \in \mathbb{R}\}$.*

The range is $\{g(x) \mid g(x) > -4\}$.

The horizontal asymptote is y = -4

The x-intercept is approximately (1.9, 0).

The y-intercept is (0, 11).

18. *The domain is $\{x \mid x \in \mathbb{R}\}$.*

The range is $\{h(x) \mid h(x) < -5\}$.

The horizontal asymptote is y = -5

There is no x-intercept.

The y-intercept is (0, -8).

19. *The domain is $\{x \mid x \in \mathbb{R}\}$.*

The range is $\{g(x) \mid g(x) > 0\}$.

The horizontal asymptote is y = 0.

There is no x-intercept.

The y-intercept is (0, 6).

20. *The domain is $\{x \mid x \in \mathbb{R}\}$.*

The range is $\{g(x) \mid g(x) < 7\}$.

The horizontal asymptote is y = 7.

The x-intercept is at approximately (-0.92, 0)

The y-intercept is at approximately (0, 3.8).

24. Use the following functions to answer the questions below.

$f(x) = 3(x - 5) + 1$ $\quad g(x) = 3^{x-5} + 1$ $\quad h(x) = \frac{3}{x-5} + 1$

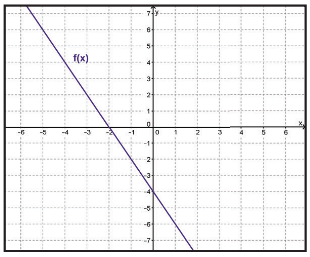
a. Identify and compare the domain of the three functions (use interval notation).
The domain for both f(x) and g(x) is (–∞, ∞), while the domain for h(x) is (–∞, 5) or (5, ∞).

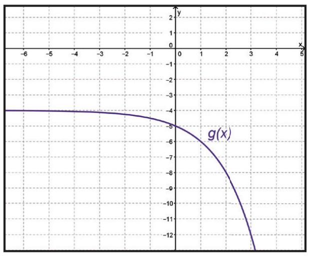
b. Identify and compare the range of the three functions (use interval notation).
The range of all three functions is different. The range of f(x) is (–∞, ∞), g(x) is (1, ∞), and h(x) is (–∞, 1) or (1, ∞).

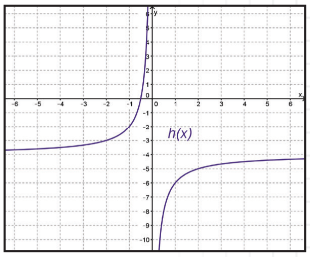
c. Identify and compare the asymptotes, if they exist, of the three functions.
See margin.

25. Use the following functions and their graphs to answer the questions below.

$f(x) = -2(x) - 4$ $\quad g(x) = -2^x - 4$ $\quad h(x) = \frac{-2}{x} - 4$







a. Identify and compare the domain of the three functions (use set-builder notation).
The domain for both f(x) and g(x) is {x | x ∈ ℝ}. The domain for h(x) is {x | x ∈ ℝ, x ≠ 0}.

b. Identify and compare the range of the three functions (use set-builder notation).
See margin.

c. Identify and compare the asymptotes, if they exist, of the three functions.
See margin.

24c. *Because f(x) is linear, it has no asymptotes. The function g(x) is exponential, so it has a horizontal asymptote at y = 1. The rational function h(x) also has a horizontal asymptote at y = 1, and has a vertical asymptote at x = 5.*

25b. *The range of all three functions is different. The range of f(x) is {f(x) | f(x) ∈ ℝ}. The function g(x) has a range of {g(x) | g(x) < -4}, while h(x) has a range of {h(x) | h(x) ∈ ℝ, h(x) ≠ -4}.*

25c. *Because f(x) is linear, it has no asymptotes. The function g(x) is exponential, so it has a horizontal asymptote at y = -4. The rational function h(x) also has a horizontal asymptote at y =-4, and has a vertical asymptote at x = 0.*