TEKS

AR.3A Compare and contrast the key attributes, including domain, range, maxima, minima, and intercepts, of a set of functions such as a set comprised of a linear, a quadratic, and an exponential function or a set comprised of an absolute value, a quadratic, and a square root function tabularly, graphically, and symbolically.

AR.7A Represent domain and range of a function using interval notation, inequalities, and set (builder) notation.

MATHEMATICAL PROCESS SPOTLIGHT

AR.1D Communicate mathematical ideas, reasoning, and their implications using multiple representations, including symbols, diagrams, graphs, and language as appropriate.

ELPS

3E Share information in cooperative learning interactions.

VOCABULARY

y-intercept, *x*-intercept, domain, range, rational function, asymptote

MATERIALS

- graphing calculator

2.5 Transforming and Analyzing Rational Functions

FOCUSING QUESTION How does a rational function compare to linear, quadratic, or cubic functions?

LEARNING OUTCOMES

- I can compare and contrast the key attributes of a rational function with polynomial or absolute value functions when I have the function as a table, graph, or written symbolically.
- I can represent the domain and range of a rational function in a variety of ways, including interval notation, inequalities, and set builder notation.
- I can use multiple representations, including symbols, graphs, tables, and language, to communicate mathematical ideas.

ENGAGE

Paula received her car insurance bill. The total bill for 6 months of coverage is $1,200. Her insurance company offers her several different payment plans without charging her a payment fee.

- 2 payments of $600 each
- 3 payments of $400 each
- 4 payments of $300 each
- 6 payments of $200 each

Sketch what you think a scatterplot of the amount of each payment (in dollars) versus the number of payments would look like.

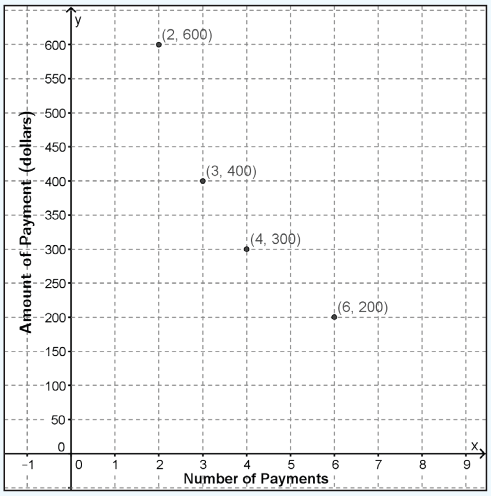
See margin.

EXPLORE

Vanessa is driving from her aunt's house in Springdale, Arkansas to her home in Little Rock, Arkansas. According to her state road map, the two cities are located 200 miles apart.

1. Complete a table like the one shown. Remember that distance, rate (speed), and time are related by the equation, $d = rt$.

ENGAGE ANSWER:



2.

SPEED (MILES PER HOUR)	TIME (HOURS)
20	10
30	$6\frac{2}{3}$
40	5
50	4
60	$3\frac{1}{3}$
70	$2\frac{6}{7}$
80	$2\frac{1}{2}$

$\Delta y = 6\frac{2}{3} - 10 = -3\frac{1}{3}$

$\Delta y = 5 - 6\frac{2}{3} = -1\frac{2}{3}$

$\Delta y = 4 - 5 = -1$

$\Delta y = 3\frac{1}{3} - 4 = -\frac{2}{3}$

$\Delta y = 2\frac{6}{7} - 3\frac{1}{3} = -\frac{10}{21}$

$\Delta y = 2\frac{1}{2} - 2\frac{6}{7} = -\frac{5}{14}$

$-1\frac{2}{3} - -3\frac{1}{3} = 1\frac{2}{3}$

$-1 - -1\frac{2}{3} = \frac{2}{3}$

$-\frac{2}{3} - -1 = \frac{1}{3}$

$-\frac{10}{21} - -\frac{2}{3} = \frac{4}{21}$

$-\frac{5}{14} - -\frac{10}{21} = \frac{5}{42}$

$\frac{2}{3} - 1\frac{2}{3} = -1$

$\frac{1}{3} - \frac{2}{3} = -\frac{1}{3}$

$\frac{4}{21} - \frac{2}{3} = -\frac{1}{7}$

$\frac{5}{42} - \frac{4}{21} = \frac{1}{14}$

SPEED (MILES PER HOUR)	TIME (HOURS)
20	10
30	$6\frac{2}{3}$
40	**5**
50	**4**
60	**$3\frac{1}{3}$**
70	**$2\frac{6}{7}$**
80	**$2\frac{1}{2}$**

2. Calculate the finite differences in the table. What patterns do you notice?
See margin.

3. Calculate the successive ratios in the table. What patterns do you notice?
See margin.

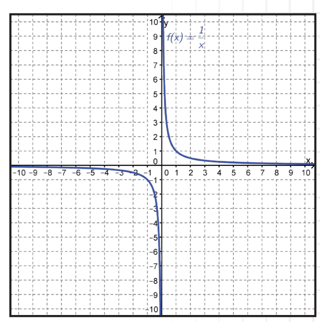
4. Does the data set appear to be modeled by a linear, quadratic, cubic, or exponential function? How do you know?
The data set appears to be modeled by none of these functions, because the finite differences and successive ratios are not constant.

5. Make a graph of the data. Describe the shape of the graph, including rates of change and key points that affect the shape of the graph.
See margin.

6. In this data set, x represents the driving speed and y represents the amount of time it took for Vanessa to reach her destination at that speed. The distance between Springdale and Little Rock is 200 miles. Use $d = rt$ to write an equation that relates distance, rate (speed), and time for this situation.
$200 = xy$

7. Solve your equation from the previous question for y and graph the curve over your data.
$y = \frac{200}{x}$ See margin for graph.

8. The rational parent function, $f(x) = \frac{1}{x}$, has the graph shown.



Use transformations of a, b, c, and d to explain how your function from the previous question is related to the rational parent function.
See margin.

2. *$\Delta x = 10$, which is constant for every consecutive pair of x-values.*

The first, second, and third differences in y are not constant and do not have any observable patterns.

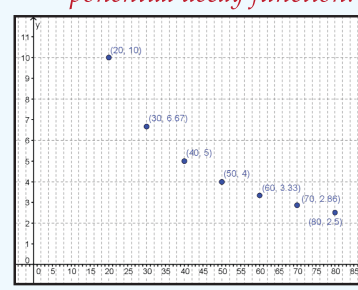
See table on page 210.

3. *$\Delta x = 10$, which is constant for every consecutive pair of x-values.*

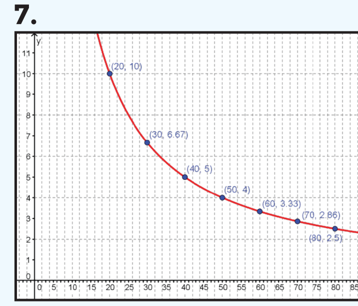
The successive ratios are not constant. However, there are patterns with the numerator and denominator for successive ratios.

See table below.

5. *The points create a curve in which y decreases as x increases. The curve appears to be exponential, but the data values in the table do not support an exponential decay function.*



7.



8. *In the function $f(x) = \frac{200}{x}$, $f(x) = 200\frac{1}{x}$ so $a = 200$. This function is the parent function vertically stretched by a factor of 200.*

3.

SPEED (MILES PER HOUR)	TIME (HOURS)	
20	10	
30	$6\frac{2}{3}$	$6\frac{2}{3} \div 10 = \frac{2}{3}$
40	**5**	$5 \div 6\frac{2}{3} = \frac{3}{4}$
50	**4**	$4 \div 5 = \frac{4}{5}$
60	**$3\frac{1}{3}$**	$3\frac{1}{3} \div 4 = \frac{5}{6}$
70	**$2\frac{6}{7}$**	$2\frac{6}{7} \div 3\frac{1}{3} = \frac{6}{7}$
80	**$2\frac{1}{2}$**	$2\frac{1}{2} \div 2\frac{6}{7} = \frac{7}{8}$

GOOD TO KNOW...

For a rational function, finite differences and successive ratios are not constant. However, there are discernable patterns that relate successive ratios of both independent and dependent function values. These patterns can be generalized for the function $y = \frac{k}{x}$.

In general, the successive ratios of dependent values of a rational function of the form $y = \frac{k}{x}$ are the reciprocals of the successive ratios of its independent values.

	x	$y = \frac{k}{x}$	
	x_1	$\frac{k}{x_1}$	
$\frac{x_2}{x_1}$	x_2	$\frac{k}{x_2}$	$\frac{k}{x_2} \div \frac{k}{x_1} = \frac{k}{x_2} \times \frac{x_1}{k} = \frac{x_1}{x_2}$
$\frac{x_3}{x_2}$	x_3	$\frac{k}{x_3}$	$\frac{k}{x_3} \div \frac{k}{x_2} = \frac{k}{x_3} \times \frac{x_2}{k} = \frac{x_2}{x_3}$
$\frac{x_4}{x_3}$	x_4	$\frac{k}{x_4}$	$\frac{k}{x_4} \div \frac{k}{x_3} = \frac{k}{x_4} \times \frac{x_3}{k} = \frac{x_3}{x_4}$
	x_{n-1}	$\frac{k}{x_{n-1}}$	
$\frac{x_n}{x_{n-1}}$	x_n	$\frac{k}{x_n}$	$\frac{k}{x_n} \div \frac{k}{x_{n-1}} = \frac{k}{x_n} \times \frac{x_{n-1}}{k} = \frac{x_{n-1}}{x_n}$

9. What are the domain and range of the function that models the data?
The domain is all real numbers except for 0, $\{x \mid x \in \mathbb{R}, x \neq 0\}$, and the range is all real numbers except for 0, $\{y \mid y \in \mathbb{R}, y \neq 0\}$.

10. What is a reasonable domain and range of the function that Vanessa could use to model the data describing her drive home? Explain your answer.
See margin.

11. A line that the graph of a function approaches, but does not cross, is called an **asymptote**. Which asymptote(s) does the graph of the rational parent function appear to have? Explain how you know they are asymptotes.
See margin.

12. What are the intercepts of the function that models the data?
There are no x-intercepts or y-intercepts.

13. What is the maximum or minimum value of the function that models the data that Vanessa collected?
There is no maximum or minimum value.

ELPS FOCUS
For the following questions, work with a partner or small group using graphing technology. As you do, be sure to share information in cooperative learning interactions.

14. Graph the functions $y = \frac{1}{x}$, $y = \frac{2}{x}$, $y = \frac{4}{x}$, $y = \frac{0.5}{x}$, and $y = -\frac{0.5}{x}$ on the same screen of your graphing calculator. Does the parameter a affect the rational parent function similarly as other functions? Explain how you know.
See margin.

15. Graph the functions $y = \frac{1}{x} + 2$, $y = \frac{1}{2x} + 2$, $y = \frac{1}{4x} + 2$, $y = \frac{1}{0.5x} + 2$, and $y = -\frac{1}{-2x} + 2$ on the same screen of your graphing calculator. Does the parameter b affect the rational parent function similarly as other functions? Explain how you know.
See margin.

16. Graph the functions $y = \frac{1}{x}$, $y = \frac{1}{x-2}$, $y = \frac{1}{x-4}$, $y = \frac{1}{x+3}$, and $y = \frac{1}{x+5}$ on the same screen of your graphing calculator. Does the parameter c affect the rational parent function similarly as other functions? Explain how you know.
See margin.

17. Graph the functions $y = \frac{1}{x}$, $y = \frac{1}{x} + 2$, $y = \frac{1}{x} + 3$, $y = \frac{1}{x} - 1$, and $y = \frac{1}{x} - 3$ on the same screen of your graphing calculator. Does the parameter d affect the rational parent function similarly as other functions? Explain how you know.
See margin.

18. Vanessa drove 200 miles at x miles per hour. If she stopped for 3 hours to visit an elderly relative, how would that change the function describing the amount of time for the trip, $f(x)$, takes as a function of her driving speed, x?
The stop adds 3 hours to the amount of time for the trip.
$f(x) = \frac{200}{x} + 3$

10. *Distance and time cannot be negative, so the domain and range should be restricted to real numbers greater than 0. Also, there are upper limits on the domain based on how fast Vanessa's car could actually travel. Let's use s miles per hour to represent this number. There are upper limits on the range based on the quotient of 200 and s, Vanessa's car's maximum speed.*

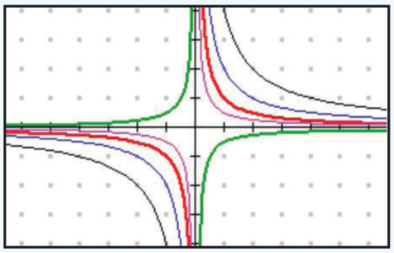
Domain: $\{x \mid 0 < x \leq s\}$ or $(0, s]$

Range: $\{y \mid 0 < y \leq \frac{200}{s}\}$ or $(0, \frac{200}{s}]$

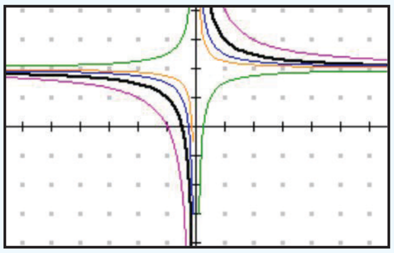
11. *There appear to be two asymptotes: a horizontal asymptote along the x-axis, $y = 0$, and a vertical asymptote along the y-axis, $x = 0$.*

When $x < 0$ and decreasing, the graph of the function gets closer and closer to the x-axis from the bottom side but does not touch it. Likewise, as $x > 0$ and increasing, the graph of the function gets closer and closer to the x-axis from the top side but does not touch it.

As x gets close to 0 from the left, the graph of the function gets closer to the y-axis but does not touch it. As x gets closer to 0 from the right, the graph of the function gets closer to the y-axis but does not touch it.



14. *Yes. $|a| > 1$ generates a vertical dilation (stretch), $0 < |a| < 1$ generates a vertical dilation (compression), and $a < 0$ generates a vertical reflection across the x-axis. These are the same transformations generated by a in other function types.*



15. *Yes. $|b| > 1$ generates a horizontal dilation (stretch) by a factor of $\frac{1}{|b|}$, $0 < |b| < 1$ generates a horizontal dilation (compression) by a factor of $\frac{1}{|b|}$, and $b < 0$ generates a horizontal reflection across the y-axis. These are the same transformations generated by b in other function types.*

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REFLECT

- Why does the graph of the rational parent function, $y = \frac{1}{x}$, have asymptotes at $x = 0$ and at $y = 0$?
  If you substitute $x = 0$ into the parent function, $y = \frac{1}{0}$, which is undefined. There is no x-value that can generate a y-value of 0.
- Where are the asymptotes for the function $g(x) = \frac{200}{x - 10} + 5$? How do you know?
  g(x) has been translated upward 5 units and to the right 10 units. The asymptotes are translated with the graph, so they should be $x = 10$ and $y = 5$.
- How do asymptotes affect the domain and range of rational functions?
  See margin.

REFLECT ANSWER:

The x-values of horizontal asymptotes are not included in the domain and the y-values of vertical asymptotes are not included in the range of rational functions.

EXPLAIN

A rational function is, by definition, a function composed of a ratio of two polynomial functions, $p(x)$ and $q(x)$.

$$r(x) = \frac{p(x)}{q(x)}$$

A polynomial function is a linear, quadratic, cubic, or higher-order function that uses an expression of one or more algebraic terms, including the sum of several terms that contain different powers of the same variable, to express the relationship between the independent variable, x, and the dependent variable, $f(x)$.

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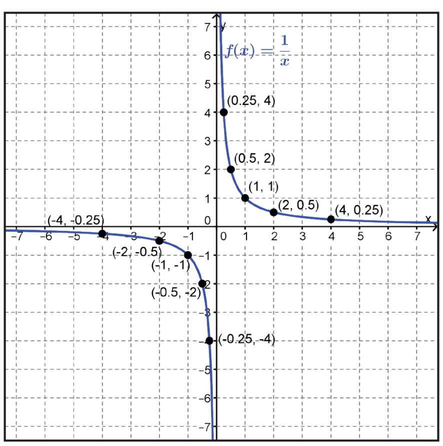
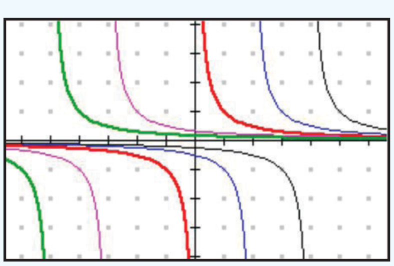
Inverse variation is a relationship between two variables such that the product of the two variables is constant. For example, If the two variables are x and y and the constant value is k, then the product $xy = k$. You can write the inverse variation relationship as a function by solving for y: $y = \frac{k}{x}$.

The simplest form of rational function is $f(x) = \frac{1}{x}$, which is also sometimes called an inverse variation function. Using the parameters a, b, c, and d, a general form of the rational function can be written: $f(x) = \frac{a}{bx - c} + d$.

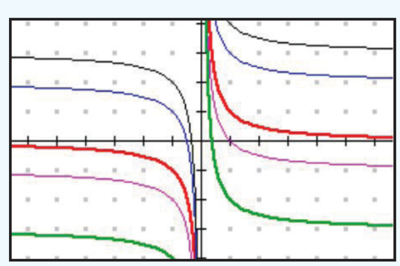
$f(x) = \frac{1}{x}$

x	$f(x)$
-4	$-\frac{1}{4}$
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
$-\frac{1}{4}$	-4
$\frac{1}{4}$	4
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
4	$\frac{1}{4}$





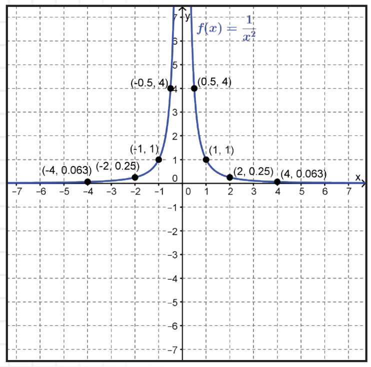
16. *Yes. $c > 0$ generates a horizontal translation $\left|\frac{c}{b}\right|$ units to the right and $c < 0$ generates a horizontal translation $\left|\frac{c}{b}\right|$ units to the left. These are the same transformations generated by c in other function types.*



17. *Yes. $d > 0$ generates a vertical translation $|d|$ units up and $d < 0$ generates a vertical translation $|d|$ units down. These are the same transformations generated by d in other function types.*

INSTRUCTIONAL HINTS

In previous sections of Chapter 2, students used a graphic organizer to record how changes a, b, c, and d influence various functions. Have students use this graphic organizer once more to compare rational functions to the other types they have learned about. What do they notice?

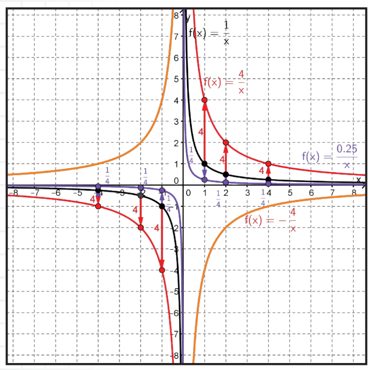


Another group of rational functions has the parent function $f(x) = \frac{1}{x^2}$, which is also sometimes called an **inverse square function**. Using the parameters a, b, c, and d, a general form of the rational function can be written: $f(x) = \frac{a}{(bx - c)^2} + d$.

CHANGES IN a

The parameter a influences the vertical dilation or a vertical reflection of the graph.

- If $|a| > 1$, then the range values (y-coordinates) of the original function are multiplied by a factor of $|a|$ in order to vertically stretch the graph.
- If $0 < |a| < 1$, then the range values (y-coordinates) of the original function are multiplied by a factor of $|a|$ in order to vertically compress the graph.
- If $a < 0$, then the graph will be reflected across the x-axis.



x	$f(x) = \frac{1}{x}$	$f(x) = \frac{4}{x}$	$f(x) = \frac{0.25}{x}$	$f(x) = -\frac{4}{x}$
-4	$-\frac{1}{4}$	-1	$-\frac{1}{16}$	1
-2	$-\frac{1}{2}$	-2	$-\frac{1}{8}$	2
-1	-1	-4	$-\frac{1}{4}$	4
0	und.	und.	und.	und.
1	1	4	$\frac{1}{4}$	-4
2	$\frac{1}{2}$	2	$\frac{1}{8}$	-2
4	$\frac{1}{4}$	1	$\frac{1}{16}$	-1

$\times 4$ $\times \frac{1}{4}$ $\times (-4)$

CHANGES IN b

The parameter b influences the horizontal stretch or compression of the graph.

- If $|b| > 1$, then the domain values (x-coordinates) of the original function are multiplied by a factor of $\frac{1}{|b|}$, which will be a multiplier that is less than 1, in order to horizontally compress the graph.
- If $0 < |b| < 1$, then the domain values (x-coordinates) of the original function are multiplied by a factor of $\frac{1}{|b|}$, which will be a multiplier that is greater than 1, in order to horizontally stretch the graph.

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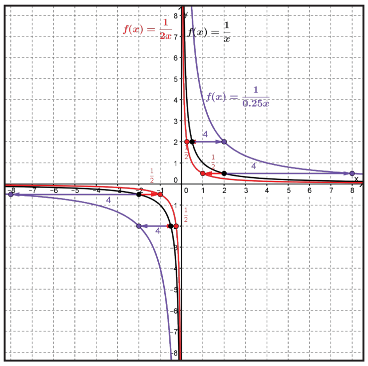


x	$f(x)=\frac{1}{x}$
-2	$-\frac{1}{2}$
$-\frac{1}{2}$	-2
0	und.
$\frac{1}{2}$	2
2	$\frac{1}{2}$

x	$f(x)=\frac{1}{2x}$
-1	$-\frac{1}{2}$
$-\frac{1}{4}$	-2
0	und.
$\frac{1}{4}$	2
1	$\frac{1}{2}$

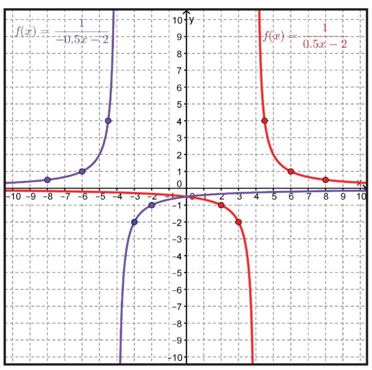
Domain (x) values are multiplied by $\frac{1}{2}$ in order to generate the same range (y) value. This multiplication results in a horizontal compression of the graph.

x	$f(x)=\frac{1}{x}$
-2	$-\frac{1}{2}$
$-\frac{1}{2}$	-2
0	und.
$\frac{1}{2}$	2
2	$\frac{1}{2}$

x	$f(x)=\frac{1}{0.25x}$
-8	$-\frac{1}{2}$
-2	-2
0	und.
2	2
8	$\frac{1}{2}$

Domain (x) values are multiplied by 4 in order to generate the same range (y) value. This multiplication results in a horizontal stretch of the graph.

The parameter b also affects the orientation of the graph. If $b < 0$, then all of the x-values will change signs and the graph will be reflected across the y-axis.



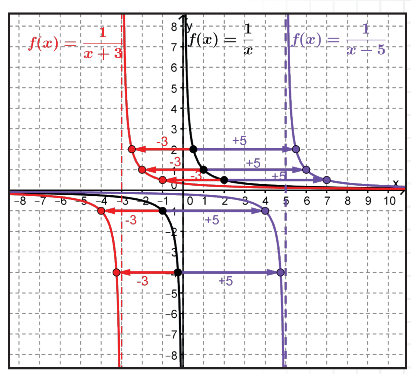
x	$f(x)=\frac{1}{0.5x-2}$
2	-1
3	-2
4.5	4
6	1
8	0.5

x	$f(x)=\frac{1}{-0.5x-2}$
-2	-1
-3	-2
-4.5	4
-6	1
-8	0.5

Domain (x) values are multiplied by −1, which changes the sign but not the magnitude of each x-value. This change results in a horizontal reflection of the graph across the y-axis.

CHANGES IN c

The parameter c influences the horizontal translation of the graph. Notice that in the general form, $f(x)=\frac{a}{bx-c}+d$, the sign in front of c is negative. That means that when reading the value of c from the equation, you should read the opposite sign from what is given in the equation.



- If $c > 0$, then the graph will translate $\left|\frac{c}{b}\right|$ units to the right.
- If $c < 0$, then the graph will translate $\left|\frac{c}{b}\right|$ units to the left.

Notice that the vertical asymptote (indicated by a dotted line), $x = 0$, is also translated horizontally with the graph. This shift will impact the domain restriction for a rational function.

x	$f(x)=\frac{1}{x}$	x	$f(x)=\frac{1}{x-5}$
-1	-1	4	-1
$-\frac{1}{4}$	-4	$4\frac{3}{4}$	-4
$\frac{1}{2}$	2	$5\frac{1}{2}$	2
1	1	6	1
2	$\frac{1}{2}$	7	$\frac{1}{2}$

Domain (x) values are increased by 5 in order to generate the same range (y) value. This addition results in a horizontal translation of the graph to the right.

x	$f(x)=\frac{1}{x}$	x	$f(x)=\frac{1}{x+3}$
-1	-1	-4	-1
$-\frac{1}{4}$	-4	$-3\frac{1}{4}$	-4
$\frac{1}{2}$	2	$-2\frac{1}{2}$	2
1	1	-2	1
2	$\frac{1}{2}$	-1	$\frac{1}{2}$

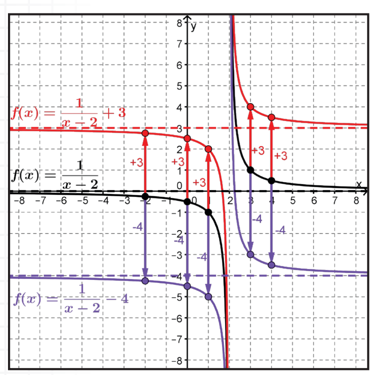
Domain (x) values are decreased by 3 in order to generate the same range (y) value. This subtraction (addition with a negative number) results in a horizontal translation of the graph to the left.

CHANGES IN d

The parameter d influences the horizontal translation of the graph.

- If $d > 0$, then the graph will translate $|d|$ units upward.
- If $d < 0$, then the graph will translate $|d|$ units downward.

Notice that the horizontal asymptote (indicated by dotted lines), $y = 0$, is also translated vertically with the graph. This shift will impact the range restriction for a rational function.



x	$f(x)=\frac{1}{x-2}$	$f(x)=\frac{1}{x-2}-4$	$f(x)=\frac{1}{x-2}+3$
-2	$-\frac{1}{4}$	$-4\frac{1}{4}$	$2\frac{3}{4}$
0	$-\frac{1}{2}$	$-4\frac{1}{2}$	$2\frac{1}{2}$
1	-1	-5	2
2	und.	und.	und.
3	1	-3	4
4	$\frac{1}{2}$	$-3\frac{1}{2}$	$3\frac{1}{2}$

−4 +3

Each of these parameters also has an effect on key attributes of a rational function.

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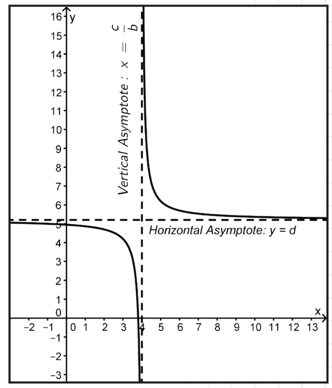
ASYMPTOTES



Each rational function in the family of $f(x) = \frac{1}{x}$ has two asymptotes: one horizontal and one vertical.

The **horizontal asymptote** is governed by vertical parameter changes. A vertical translation moves the asymptote d units and a vertical dilation does not move the asymptote.

The **vertical asymptote** is governed by horizontal parameter changes. A horizontal translation moves the asymptote c units and a horizontal dilation moves the asymptote closer or away from the x-axis by a factor of $\frac{1}{b}$.

Horizontal asymptote: $y = d$

Vertical asymptote: $x = \frac{c}{b}$

DOMAIN AND RANGE

Rational functions involve the ratio of two polynomial functions. The function in the denominator can never be equal to 0 because division by 0 is undefined. Thus, any values of x that cause the denominator of a rational function to be equal to 0 are excluded from the domain.

For example, consider the function $f(x) = \frac{1}{x-2} - 4$.

In this function, $f(0) = \frac{1}{0-2} - 4 = -\frac{1}{2} - 4 = -4\frac{1}{2}$. But $f(2) = \frac{1}{2-2} - 4 = \frac{1}{0} - 4$, and you cannot divide 1 by 0. Therefore, we must exclude 2 from the domain of this function.

In the general form of a rational function, $f(x) = \frac{a}{bx-c} + d$, the domain excludes the x-value related to the vertical asymptote, $x = \frac{c}{b}$.

In terms of the range, let's look more closely at $f(x) = \frac{a}{bx-c} + d$. The only way that the function value can equal d is if $\frac{a}{bx-c} = 0$. The only way $\frac{a}{bx-c}$ can be equal to 0 is if $a = 0$. But, if $a = 0$, then the function is not a rational function; it is a constant function, $f(x) = d$. Hence, d must be excluded from the range of a rational function.

Domain: $\{x \mid x \in \mathbb{R}, x \neq \frac{c}{b}\}$
Range: $\{y \mid y \in \mathbb{R}, y \neq d\}$

y-INTERCEPT

The y-intercept of any function is the point $(0, y)$, or the point where the graph intersects the y-axis. For a rational function, you can calculate the y-intercept by substituting $x = 0$ into the general form, $y = \frac{a}{bx-c} + d$.

INSTRUCTIONAL HINTS

Have students note asymptotes on graphs and tables using different colored highlighters or map pencils.

ADDITIONAL EXAMPLES

What transformations of the rational parent function, $f(x) = \frac{1}{x}$, will result in the graphs of the rational functions below?

1. $h(x) = \frac{2}{-3x + 5} - 6.5$

The graph of h(x) is produced by transforming the rational parent function f(x) by vertically stretching its graph by a factor of 2, horizontally compressing its graph by a factor of $\frac{1}{3}$, reflecting its graph over the y-axis, and translating its graph $\frac{5}{3}$ units to the right and 6.5 units down.

2. $j(x) = \frac{-0.5}{-6x - 18} + 14$

The graph of j(x) is produced by transforming the rational parent function f(x) by vertically compressing its graph by a factor of $\frac{1}{2}$, horizontally compressing its graph by a factor of $\frac{1}{6}$, reflecting its graph over the x-axis and y-axis, and translating its graph 3 units to the left and 14 units up.

3. $k(x) = \frac{-1}{0.25x} - 2$

The graph of k(x) is produced by transforming the rational parent function f(x) by horizontally stretching its graph by a factor of 4, reflecting its graph over the x-axis, and translating its graph 2 units down.

$$y = \frac{a}{bx - c} + d$$
$$y = \frac{a}{b(0) - c} + d$$
$$y = \frac{a}{-c} + d$$

The coordinates of the y-intercept are $(0, \frac{a}{-c} + d)$.

The parameters a, c, and d each influence the y-intercept of a rational function. The parameter d vertically shifts the entire graph, including the y-intercept. The parameter c horizontally shifts the entire graph, including the y-intercept. The parameter a vertically dilates or reflects the entire graph, including the y-intercept. Each of these transformations affects where the graph intersects the y-axis.

KEY ATTRIBUTES OF RATIONAL FUNCTIONS, $f(x) = \frac{1}{x}$

A rational function has several important key attributes:

- The domain of a rational function is all real numbers, excluding the vertical asymptote. $\{x \mid x \in \mathbb{R}, x \neq \frac{c}{b}\}$
- The range of a rational function is all real numbers, excluding the horizontal asymptote. $\{y \mid y \in \mathbb{R}, y \neq d\}$
- The horizontal asymptote of a function in the family of $f(x) = \frac{1}{x}$ is $y = d$.
- The vertical asymptote of a function in the family of $f(x) = \frac{1}{x}$ is $x = \frac{c}{b}$.
- A rational function in the family of $f(x) = \frac{1}{x}$ has at most one x-intercept. If it exists, its coordinates are $\left(\frac{cd - a}{bd}, 0\right)$. The values of b and d may not equal 0. When $b = 0$, the function does not exist since there is no x-term in the function. When $d = 0$, the x-axis is an asymptote and the x-intercept does not exist.
- A rational function has at most one y-intercept at $(0, \frac{a}{-c} + d)$. When $c = 0$, the y-axis is an asymptote and the y-intercept does not exist.

EXAMPLE 1

What transformations of the rational parent function, $f(x) = \frac{1}{x}$, will result in the graph of the rational function $g(x) = -\frac{3}{x-2} + 1.5$?

STEP 1 Rewrite the equation of $g(x)$ in general form $y = \frac{a}{bx - c} + d$ to determine the values of the parameters a, b, c and d.

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$g(x) = -\frac{3}{x-2} + 1.5$

$g(x) = \frac{-3}{1x-2} + 1.5$

Therefore, $a = -3$, $b = 1$, $c = 2$, and $d = 1.5$

STEP 2 **Use the values of the parameters to describe the transformations of the rational parent function $f(x) = \frac{1}{x}$ that are necessary to produce $g(x) = -\frac{3}{x-2} + 1.5$.**

$a = -3$, so $|a| > 0$. The range values (y-coordinates) of the rational parent function are multiplied by a factor of 3 in order to vertically stretch the graph of the function. Additionally, since a is negative, the graph is reflected across the x-axis.

$b = 1$, so there is no horizontal dilation of the rational parent function. Since b is positive, there is no reflection across the y-axis.

$c = 2$, so $c > 0$. The graph of the rational parent function will translate $\left|\frac{2}{1}\right| = 2$ units to the right.

$d = 1.5$, so $d > 0$. The graph of the rational parent function will translate $|1.5| = 1.5$ units up.

The graph of g(x) is produced by transforming the rational parent function f(x) by vertically stretching its graph by a factor of 3, reflecting its graph over the x-axis, and translating its graph 2 units to the right and 1.5 units up.

YOU TRY IT! #1

What transformations of the rational parent function, $f(x) = \frac{1}{x}$, will result in the graph of the rational function $f(x) = \frac{10}{2x+5} - 1$?

See margin.

YOU TRY IT! #1 ANSWER:

The graph of $f(x) = \frac{10}{2x+5} - 1$ is produced by transforming the rational parent function by vertically stretching its graph by a factor of ten, horizontally compressing its graph by a factor of one half, and translating its graph five-halves units to the left and one unit down. Since both a and b are greater than zero, there is no reflection over the x- or y-axis.

ADDITIONAL EXAMPLES

Identify the key attributes of the functions below, including domain and range (including asymptotes), x-intercept, and y-intercept. Write the domain and range in set builder notation.

1. $y = \frac{-2}{5x + 10} - 3$

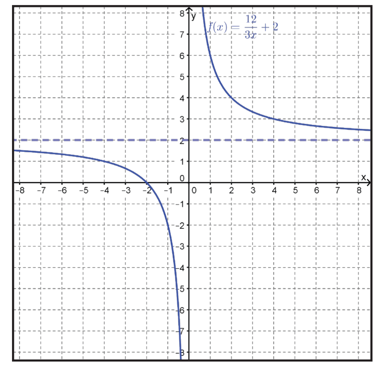
Domain: $\{x \mid x \in \mathbb{R}, x \neq -2\}$
Range: $\{y \mid y \in \mathbb{R}, y \neq -3\}$
Asymptotes: $x = -2$ and $y = -3$
x-intercept: $(-2.13, 0)$
y-intercept: $(0, -3.2)$

2. $y = \frac{0.5}{0.25x - 4} + 6$

Domain: $\{x \mid x \in \mathbb{R}, x \neq 16\}$
Range: $\{y \mid y \in \mathbb{R}, y \neq 6\}$
Asymptotes: $x = 16$ and $y = 6$
x-intercept: $(15.67, 0)$
y-intercept: $(0, 5.875)$

EXAMPLE 2

Identify the key attributes of $y = \frac{12}{3x} + 2$, including domain and range (including asymptotes), x-intercept, and y-intercept. Write the domain and range in set builder notation.



STEP 1 **Determine the domain and range of $y = \frac{12}{3x} + 2$.**

From the graph, we can see the domain contains all real values of x, excluding $x = 0$. The domain for rational functions excludes the value of $\frac{c}{b}$, which in this equation is $\frac{0}{3} = 0$. Written in set builder notation, the domain is $\{x \mid x \in \mathbb{R}, x \neq 0\}$. This relates to the vertical asymptote, $x = \frac{0}{3} = 0$.

From the equation, the value of $d = 2$ indicates that the graph of the rational function would have a horizontal asymptote. This is confirmed by the provided graph. Since $d = 2$, the range of the function contains all real values of y excluding 2. This is written in set builder notation as $\{y \mid y \in \mathbb{R}, x \neq 2\}$.

STEP 2 **Determine if the function has an x-intercept.**

The coordinates of the x-intercept of a rational function are $\left(\frac{cd - a}{bd}, 0\right)$. Given the parameters in the equation, the x-intercept $= \left(\frac{cd - a}{bd}, 0\right) = \left(\frac{0(2) - 12}{3(2)}, 0\right) = \left(\frac{-12}{6}, 0\right) = (-2, 0)$. This is confirmed by the graph.

STEP 3 **Determine if the function has a y-intercept.**

The coordinates of the y-intercept of a rational function are $(0, \frac{a}{-c} + d)$, which in this equation, would be $(0, \frac{12}{0} + 2)$. But division by 0 is undefined. So there is no y-intercept and this is confirmed by the graph.

YOU TRY IT! #2

Identify the key attributes of $y = \frac{1}{x - 2} + 5$, including domain and range (including asymptotes), x-intercept, and y-intercept. Write the domain and range in set builder notation.

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x	y
-3	4.8
-2	4.75
-1	4.67
0	4.5
1	4
2	ERROR
3	6
4	5.5

See margin.

EXAMPLE 3

Identify and compare the x- and y-intercepts of $f(x) = 3x + 2$ and $g(x) = \frac{1}{3x+2}$.

x	$f(x)$	$g(x)$
$-1\frac{2}{3}$	−3	$-\frac{1}{3}$
−1	−1	−1
$-\frac{2}{3}$	0	ERROR
$-\frac{1}{3}$	1	1
0	2	$\frac{1}{2}$
$\frac{1}{3}$	3	$\frac{1}{3}$
$\frac{2}{3}$	4	$\frac{1}{4}$
1	5	$\frac{1}{5}$

STEP 1 **Determine the x-intercept(s) of $f(x)$.**

Since $f(x)$ is a linear function, $f(x)$ has one x-intercept, where $y = 0$.

$$0 = 3x + 2$$
$$-2 = 3x$$
$$x = -\frac{2}{3}$$

The x-intercept of $f(x)$ is $(-\frac{2}{3}, 0)$.

YOU TRY IT! #2 ANSWER:

The domain is $\{x \mid x \in \mathbb{R}, x \neq 2\}$. The range is $\{y \mid y \in \mathbb{R}, y \neq 5\}$. The vertical asymptote is $x = 2$ and the horizontal asymptote is $y = 5$. The coordinates of the x-intercept of the function are $(1.8, 0)$. The coordinates of the y-intercept are $(0, \frac{9}{2})$.

ADDITIONAL EXAMPLE

Identify and compare the x- and y-intercepts of $h(x) = -0.5x + 3$ and $k(x) = \frac{1}{-0.5x+3}$.

The x-intercept of $h(x)$ is $(6, 0)$. $k(x)$ does not have an x-intercept.

The y-intercept of $h(x)$ is $(0, 3)$. The y-intercept of $k(x)$ is $(0, \frac{1}{3})$.

ADDITIONAL EXAMPLE

Identify and compare the x- and y-intercepts of $p(x) = 3(0.25x + 4)^2 - 27$ and $q(x) = \frac{3}{0.25x + 4} - 27$.

The x-intercepts of $p(x)$ are $(-4, 0)$ and $(-28, 0)$.
The x-intercept of $q(x)$ is $(-15.56, 0)$.

The y-intercept of $p(x)$ is $(0, 21)$.
The y-intercept of $q(x)$ is $(0, -26.25)$.

STEP 2 **Determine the x-intercept(s) of $g(x)$.**

Since $g(x)$ is a rational function, the x-intercept is at $\left(\frac{cd - a}{bd}, 0\right) = \left(\frac{-2(0) - 1}{3(0)}, 0\right) = \left(\frac{-1}{0}, 0\right)$. But division by zero is undefined, and so there is no x-intercept.

STEP 3 **Determine the y-intercept, of the function $f(x)$.**

Since $y = 3x + 2$ is a linear function, its y-intercept is where $x = 0$.

$3(0) + 2 = 2$

The coordinates of the y-intercept of $f(x)$ are $(0, 2)$.

STEP 4 **Determine the coordinates of the y-intercept of $g(x)$.**

Since $y = \frac{1}{3x + 2}$ is a rational function, its y-intercept is located at $\left(0, \frac{a}{-c} + d\right)$.

$\left(0, \frac{1}{2} + 0\right) = \left(0, \frac{1}{2}\right)$.

The coordinates of the y-intercept of $g(x)$ are $\left(0, \frac{1}{2}\right)$.

Since $f(x)$ is a linear function, it has only one x-intercept at $\left(-\frac{2}{3}, 0\right)$. The rational function $g(x)$ has no x-intercept.

The y-intercept of the linear function $f(x)$ is $(0, 2)$. The y-intercept of the quadratic function $g(x)$ is $\left(0, \frac{1}{2}\right)$. The y-intercepts of the linear and rational functions, as all points, are reciprocals because the functions are the reciprocal of each other. The exception is $\left(-\frac{2}{3}, 0\right)$ because the reciprocal of zero is undefined.

YOU TRY IT! #3

Identify and compare the y-intercept(s) of $f(x) = 0.1(3x - 5) - 1$ and $g(x) = -\frac{0.1}{3x - 5} - 1$

See margin.

YOU TRY IT! #3 ANSWER:

The y-intercept of $f(x)$ is $(0, -1.5)$. The y-intercept of $g(x)$ is $(0, -0.98)$. Even though the parameters a, b, c, and d are the same for both functions, their y-intercepts are different. It is interesting to note that finding the y-intercepts is different in one particular way: for linear functions, the y-intercept is $(0, -ac + d)$ and for rational functions, the y-intercept is $(0, \frac{a}{-c} + d)$. Disregarding the addition of d, notice that one has the product of a and -c and the other has their quotient.

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EXAMPLE 4

Identify and compare the domains and ranges (with asymptotes, if any) of $f(x) = 7(x-2)^2 + 10$, $g(x) = 7(x-2)^3 + 10$, and $h(x) = \frac{7}{x-2} + 10$. Write the domain and range of each function as inequalities and in set builder notation.

STEP 1 **Determine the domain and range of $f(x)$.**

Since $f(x)$ is a quadratic function, the domain contains all real values of x. Written as an inequality, $-\infty < x < \infty$. The same information about the domain written in set builder notation is $\{x \mid x \in \mathbb{R}\}$. There is no vertical asymptote.

Using the equation, the value of a indicates that the parabola opens upward. The only other parameter that affects the range of a quadratic function is d. Since $d = 10$, the range of this quadratic function contains all real values of $f(x)$ that are greater than or equal to ten. As an inequality, this is written $10 \leq f(x) < \infty$. The same information about the range written in set builder notation is $\{f(x) \mid f(x) \geq 10\}$. There is no horizontal asymptote.

STEP 2 **Determine the domain and range of $g(x)$.**

Since $g(x)$ is a cubic function, its domain contains all real values of x. Written as an inequality, $-\infty < x < \infty$. The same information about the domain written in set builder notation is $\{x \mid x \in \mathbb{R}\}$.

Since $g(x)$ is a cubic function, its range contains all real values of $g(x)$. Written as an inequality, $-\infty < g(x) < \infty$. The same information about the range written in set builder notation is $\{g(x) \mid g(x) \in \mathbb{R}\}$.

A cubic function has neither a vertical nor a horizontal asymptote.

STEP 3 **Determine the domain and range of $h(x)$.**

Since $h(x)$ is a rational function, the domain is all real numbers, excluding the vertical asymptote, $\{x \mid x\ ,\ x \in \mathbb{R} \neq \frac{c}{b}\}$. In this function, $\frac{c}{b} = \frac{2}{1} = 2$. Written as an inequality, $-\infty < x < 2$ or $2 < x < \infty$. The same information about the domain written in set builder notation is $\{x \mid x \in \mathbb{R}, x \neq 2\}$.

The range of a rational function is all real numbers, excluding the horizontal asymptote. $\{y \mid y \in \mathbb{R}, y \neq d\}$, In this function, $d = 10$. Thus the range of this rational function contains all real values of $h(x)$ that are not equal to ten. As an inequality, this is written $-\infty < y < 10$ or $10 < y < \infty$. The same information about the range written in set builder notation is $\{y \mid y \in \mathbb{R}, y \neq 10\}$.

ADDITIONAL EXAMPLE

Identify and compare the domains and ranges (with asymptotes, if any) of the given functions.

$p(x) = -0.4(5x - 10) + 12$

$q(x) = -0.4(5x - 10)^3 + 12$

$r(x) = \frac{-0.4}{5x - 10} + 12$

Write the domain and range of each function as inequalities and in set builder notation.

The domains of p(x) and q(x) both contain all real values of x. The domain of r(x) contains all real numbers except x=2, the x-value related to the vertical asymptote. The ranges of p(x) and q(x) contain all real values of y. The range of r(x) contains all real numbers excluding y=12, the y-value related to the horizontal asymptote.

	NOTATION	$p(x) = -0.4(5x - 10) + 12$	$q(x) = -0.4(5x - 10)^3 + 12$	$r(x) = \frac{-0.4}{5x - 10} + 12$
DOMAIN	INEQUALITY	$-\infty < x < \infty$	$-\infty < x < \infty$	$-\infty < x < 2$ or $2 < x < \infty$
	SET BUILDER	$\{x \mid x \in \mathbb{R}\}$	$\{x \mid x \in \mathbb{R}\}$	$\{x \mid x \in \mathbb{R}, x \neq 2\}$
RANGE	INEQUALITY	$-\infty < p(x) < \infty$	$-\infty < q(x) < \infty$	$-\infty < r(x) < 12$ or $12 < y < \infty$
	SET BUILDER	$\{p(x) \mid p(x) \in \mathbb{R}\}$	$\{q(x) \mid q(x) \in \mathbb{R}\}$	$\{r(x) \mid r(x) \in \mathbb{R}, r(x) \neq 12\}$
ASYMPTOTE	VERTICAL	NONE	NONE	$x = 2$
	HORIZONTAL	NONE	NONE	$y = 12$

STEP 4 **Compare the domains and ranges of $f(x)$, $g(x)$ and $h(x)$.**

The domains of first two functions are the same because both $f(x)$ and $g(x)$ are polynomials, one of degree two and the other of degree three, whereas $h(x)$ is a rational function and its domain excludes the x-value related to its vertical asymptote. The range of the quadratic function $f(x)$ does not contain all real numbers but is restricted to only those real numbers greater than or equal to ten, whereas the cubic function $g(x)$ has no restrictions. The range of the rational function $h(x)$ excludes the y-value related to its horizontal asymptote.

	NOTATION	$f(x) = 7(x-2)^2 + 10$	$g(x) = 7(x-2)^3 + 10$	$h(x) = \frac{7}{x-2} + 10$
DOMAIN	INEQUALITY	$-\infty < x < \infty$	$-\infty < x < \infty$	$-\infty < x < 2$ or $2 < x < \infty$
	SET BUILDER	$\{x \mid x \in \mathbb{R}\}$	$\{x \mid x \in \mathbb{R}\}$	$\{x \mid x \in \mathbb{R}, x \neq 2\}$
RANGE	INEQUALITY	$10 \leq f(x) < \infty$	$-\infty < g(x) < \infty$	$-\infty < y < 10$ or $10 < y < \infty$
	SET BUILDER	$\{f(x) \mid f(x) \geq 10\}$	$\{g(x) \mid g(x) \in \mathbb{R}\}$	$\{y \mid y \in \mathbb{R}, y \neq 10\}$
ASYMPTOTE	VERTICAL	NONE	NONE	$x = 2$
	HORIZONTAL	NONE	NONE	$y = 10$

YOU TRY IT! #4

Identify and compare the domains and ranges (with asymptotes, if any) of $f(x) = (x+4)^2$, $g(x) = |x+4|$, and $h(x) = \frac{1}{x+4}$. Write the domain and range of each function as inequalities and in set builder notation.

See margin.

YOU TRY IT! #4 ANSWER:

The domains of $f(x)$ and $g(x)$ both contain all real values of x. The domain of $h(x)$ contains all real numbers except $x = \frac{c}{b} = -4$, the x-value related to the vertical asymptote. The ranges of $f(x)$ and $g(x)$ contain all real numbers greater than or equal to zero (the minimum for both functions is (-4, 0). The range of $h(x)$ contains all real numbers excluding $y = d = 0$, the y-value related to the horizontal asymptote.

	NOTATION	$f(x) = (x+4)^2$	$g(x) = \lvert x+4 \rvert$	$h(x) = \frac{1}{x+4}$
DOMAIN	INEQUALITY	$-\infty < x < \infty$	$-\infty < x < \infty$	$-\infty < x < -4$ or $-4 < x < \infty$
	SET BUILDER	$\{x \mid x \in \mathbb{R}\}$	$\{x \mid x \in \mathbb{R}\}$	$\{x \mid x \in \mathbb{R}, x \neq -4\}$
RANGE	INEQUALITY	$0 \leq y < \infty$	$0 < y < \infty$	$-\infty < y < 0$ or $0 < y < \infty$
	SET BUILDER	$\{f(x) \mid f(x) \geq 0\}$	$\{g(x) \mid g(x) \geq 0\}$	$\{h(x) \mid h(x) \in \mathbb{R}, y \neq 0\}$
ASYMPTOTE	VERTICAL	NONE	NONE	$x = -4$
	HORIZONTAL	NONE	NONE	$y = 0$

PRACTICE/HOMEWORK

For questions 1 – 8, identify the transformations of the rational parent function $f(x) = \frac{1}{x}$ that will result in the graph of each rational function.

1. $f(x) = \frac{1}{(x+1)}$
See margin.

2. $f(x) = \frac{4}{x} - 2$
See margin.

3. $f(x) = \frac{1}{3x-5}$
See margin.

4. $f(x) = \frac{0.3}{2x} + 1$
See margin.

5. $f(x) = \frac{1}{-0.5x}$
See margin.

6. $f(x) = \frac{-0.5}{x-2}$
See margin.

7. $f(x) = \frac{1}{x} - 5$
See margin.

8. $f(x) = \frac{10}{x} + 1$
See margin.

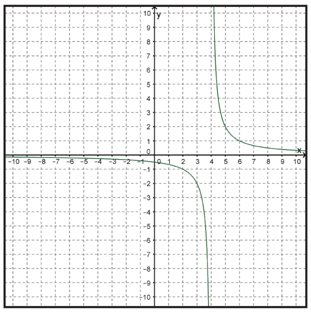
For questions 9 – 13, determine the equation of the horizontal and vertical asymptote of the given rational functions.

9. $f(x) = \frac{3}{2x+1}$
See margin.

10. $f(x) = \frac{-2}{x-1} - 4$
See margin.

11. $f(x) = \frac{5}{x+7} + 3$
See margin.

12.



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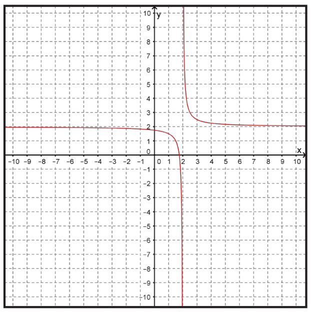
13.



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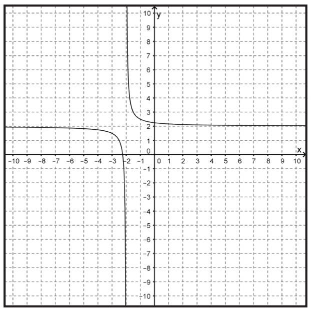
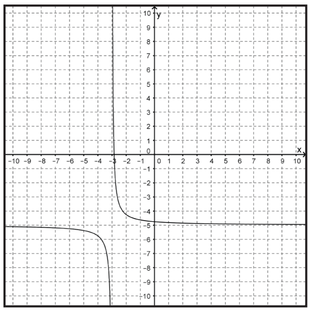
For questions 14 – 17, determine the domain and range of each rational function represented. Write the domain and range in set builder notation.

14.



Domain $\{x \mid x \in \mathbb{R}, x \neq -2\}$
Range $\{y \mid y \in \mathbb{R}, y \neq 2\}$

15.



Domain $\{x \mid x \in \mathbb{R}, x \neq -3\}$
Range $\{y \mid y \in \mathbb{R}, y \neq -5\}$

16. $f(x) = \frac{8}{3x-12}$
Domain $\{x \mid x \in \mathbb{R}, x \neq 4\}$
Range $\{y \mid y \in \mathbb{R}, y \neq 0\}$

17. $f(x) = \frac{6}{x+3}$
Domain $\{x \mid x \in \mathbb{R}, x \neq -3\}$
Range $\{y \mid y \in \mathbb{R}, y \neq 0\}$

1. *Horizontal shift left 1 unit.*

2. *Vertical stretch by a factor of 4, vertical shift down 2 units.*

3. *Horizontal compression by a factor of $\frac{1}{3}$, horizontal shift right $\frac{5}{3}$ units.*

4. *Vertical compression by a factor of 0.3, horizontal compression by a factor of $\frac{1}{2}$, vertical shift up 1 unit.*

5. *Horizontal stretch by a factor of 2, horizontal reflection across the y-axis.*

6. *Vertical compression by a factor of 0.5, vertical reflection across the x-axis, horizontal shift right 2 units.*

7. *Vertical shift down 5 units.*

8. *Vertical stretch by a factor of 10, vertical shift up 1 unit.*

9. *Horizontal asymptote: $y = 0$*

Vertical asymptote: $x = -\frac{1}{2}$

10. *Horizontal asymptote: $y = -4$*

Vertical asymptote: $x = 1$

11. *Horizontal asymptote: $y = 3$*

Vertical asymptote: $x = -7$

12. *Horizontal asymptote: $y = 0$*

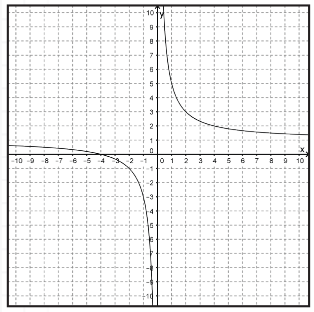
Vertical asymptote: $x = 4$

13. *Horizontal asymptote: $y = 2$*

Vertical asymptote: $x = 2$

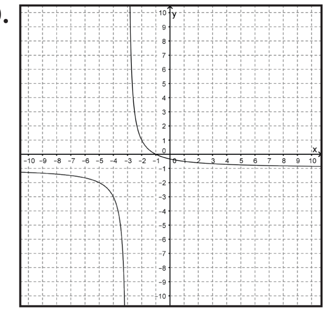
For each rational function in questions 18 – 21, determine the x-intercept and y-intercept. If no x-intercept or y-intercept exists, write none.

18.



x-intercept: (-4, 0)
y-intercept: none

19.



x-intercept: (-1, 0)
y-intercept: $\left(0, -\frac{1}{3}\right)$

20. $f(x) = \frac{-6}{2x+1} + 2$

x-intercept: (1, 0)
y-intercept: (0, -4)

21. $f(x) = \frac{0.5}{x-2} - 2$

x-intercept: (2.25, 0)
y-intercept: (0. -2.25)

22. Determine the attributes of each function and complete the table below.

	$f(x) = 2x + 1$	$f(x) = (2x+1)^2$	$f(x) = \frac{1}{2x+1}$
DOMAIN (INEQUALITY)	$-\infty < x < \infty$	$-\infty < x < \infty$	$-\infty < x < -0.5$ or $-0.5 < x < \infty$
DOMAIN (SET BUILDER NOTATION)	$\{x \mid x \in \mathbb{R}\}$	$\{x \mid x \in \mathbb{R}\}$	$\{x \mid x \in \mathbb{R}, x \neq -0.5\}$
RANGE (INEQUALITY)	$-\infty < y < \infty$	$y \geq 0$	$-\infty < y < 0$ or $0 < y < \infty$
RANGE (SET BUILDER NOTATION)	$\{y \mid y \in \mathbb{R}\}$	$\{y \mid y \geq 0\}$	$\{y \mid y \in \mathbb{R}, y \neq 0\}$
HORIZONTAL ASYMPTOTE	none	none	$y = 0$
VERTICAL ASYMPTOTE	none	none	$x = -0.5$
x-INTERCEPT	(-0.5, 0)	(-0.5, 0)	none
y-INTERCEPT	(0, 1)	(0, 1)	(0, 1)

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