Transforming and Analyzing Absolute Value Functions

2.4

FOCUSING QUESTION How are the graphs of absolute value functions similar to graphs of linear functions or quadratic functions?

LEARNING OUTCOMES

- I can compare and contrast the key attributes of an absolute value function with polynomial functions when I have the function as a table, graph, or written symbolically.
- I can represent the domain and range of an absolute value function in a variety of ways, including interval notation, inequalities, and set builder notation.
- I can use multiple representations, including symbols, graphs, tables, and language to communicate mathematical ideas.

ENGAGE

Kolomoki Mounds, Georgia

The Gregorian calendar, introduced by Pope Gregory XIII in 1582, is used in most countries to indicate the year. There is no year 0, and years are indicated as BCE (Before the Common Era) or CE (Common Era). The dates in which different empires existed are shown below.

- Roman Empire, 31 BCE to 295 CE
- Han Dynasty (China), 206 BCE to 220 CE
- Hopewell Culture (North America), 200 BCE to 500 CE
- Kingdom of Ghana, 750 CE to 1078 CE

List the empires in order of shortest duration to longest duration.

Roman Empire, Kingdom of Ghana, Han Dynasty (China), Hopewell Culture (North America)

EXPLORE

Noah and his family were traveling to visit family. Along their journey, Noah knew they would drive through Junction, Texas. Noah made a table of the time and the distance they were from the main exit for Junction.

TEKS

AR.3A Compare and contrast the key attributes, including domain, range, maxima, minima, and intercepts, of a set of functions such as a set comprised of a linear, a quadratic, and an exponential function or a set comprised of an absolute value, a quadratic, and a square root function tabularly, graphically, and symbolically.

AR.7A Represent domain and range of a function using interval notation, inequalities, and set (builder) notation.

MATHEMATICAL PROCESS SPOTLIGHT

AR.1D Communicate mathematical ideas, reasoning, and their implications using multiple representations, including symbols, diagrams, graphs, and language as appropriate.

ELPS

1F Use accessible language and learn new and essential language in the process.

VOCABULARY

absolute value, translation, reflection, dilation, slope, y-intercept, x-intercept, domain, range, maximum, minimum

MATERIALS

- graphing calculator

DIFFERENTIATING INSTRUCTION

Provide students that need extra assistance with a number line marked off in hundreds of years with 0 at the middle. Ask students to represent the years as integers along the number line.

1. *Δx = 6, which is constant for every consecutive pair of x-values.*

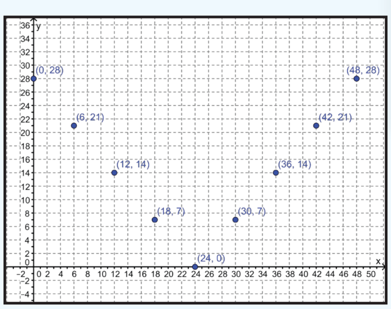
The first four differences in y-values are −7 but the next four differences in y-values are +7.

TIME, x (MINUTES)	DISTANCE, y (MILES)
0	28
6	21
12	14
18	7
24	0
30	7
36	14
42	21
48	28

Δy = 21 – 28 = –7
Δy = 14 – 21 = –7
Δy = 7 – 14 = –7
Δy = 0 – 7 = –7
Δy = 7 – 0 = +7
Δy = 14 – 7 = +7
Δy = 21 – 14 = +7
Δy = 28 – 21 = +7

2. *None of these functions would model the data set directly. The first differences are constant in two sets, −7 and +7. The second differences and third differences are not constant. Successive ratios are not constant.*

3. *The data appear to be V-shaped. There is a constant rate of change of $-\frac{7}{6}$ until (24, 0), when the constant rate of change becomes $+\frac{7}{6}$. It appears as though two lines that intersect at (24, 0) will model the data.*



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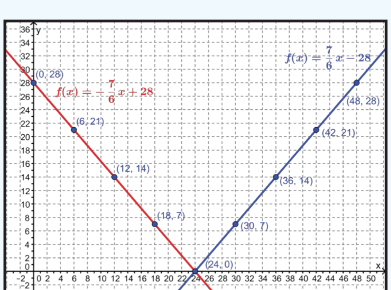


TIME, x (MINUTES)	DISTANCE, y (MILES)
0	28
6	21
12	14
18	7
24	0
30	7
36	14
42	21
48	28

Nicholas Henderson, ***Junction, Texas, City Limits,*** Flickr

1. Calculate the first differences in the table. What do you notice about the first differences?
See margin.

2. Does the data set appear to be modeled by a linear, quadratic, cubic, or exponential function? How do you know?
See margin.

3. Make a graph of the data. Describe the shape of the graph, including rates of change and key points that affect the shape of the graph.
See margin.

4. Use the graph and table to divide the data set into two different parts that could each be modeled with a different function. Write the functions and graph them over your data set.
See margin.

5. For your set of functions, identify the domain for which your function models the data.
For $f(x) = -\frac{7}{6}x + 28$, the domain is [0, 24] and for $f(x) = \frac{7}{6}x - 28$, the domain is [24, 48].

6. Factor out −1 from all terms of the second function you wrote. How does the new function rule compare to the function rule you wrote for the first function?
$f(x) = \frac{7}{6}x - 28 = -(-\frac{7}{6}x + 28)$, which is the same as the first function with a = −1.

7. What transformation does $a = -1$ represent?
a = −1 represents a reflection across the x-axis.

8. What do the following pairs of numbers have in common: |6| and |−6|, |−5.2| and |5.2|, and |3*x*| and |−3*x*|?
Each pair contains two equivalent numbers. |6| = |−6| = 6, |−5.2| = |5.2| = 5.2, and |3x| = |−(3x)| = 3|x|.

The **absolute value** of a number, *x*, is represented by the notation |*x*|. Absolute value represents the distance between the number, *x*, and 0 on a number line.

TIME, x (MINUTES)	DISTANCE, y (MILES)
0	28
6	21
12	14
18	7
24	0
30	7
36	14
42	21
48	28

$f(x) = -\frac{7}{6}x + 28$ → (rows 0 to 24)

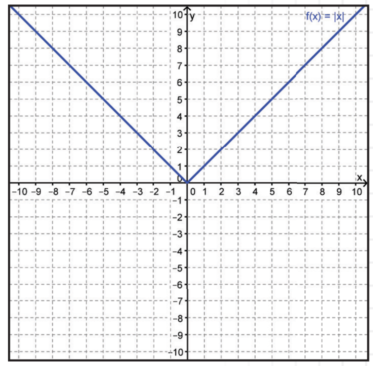
$f(x) = \frac{7}{6}x - 28$ → (rows 24 to 48)

Δy = 21 – 28 = –7
Δy = 14 – 21 = –7
Δy = 7 – 14 = –7
Δy = 0 – 7 = –7
Δy = 7 – 0 = +7
Δy = 14 – 7 = +7
Δy = 21 – 14 = +7
Δy = 28 – 21 = +7

9. How could you write your set of functions as one function?
$f(x) = \left|-\frac{7}{6}x + 28\right|$

10. Rewrite your function in the form $f(x) = a|x - c|$ by factoring out a constant and then taking its absolute value to generate the value of a.
$f(x) = \left|-\frac{7}{6}(x - 24)\right| = \frac{7}{6}|x - 24|$

11. The absolute value parent function, $f(x) = |x|$, has the graph shown.



Use transformations of a, b, c, and d to explain how your function from the previous question is related to the absolute value parent function.
See margin.

12. What are the domain and range of the function that models the data that Noah collected?
The domain is all real numbers, $\{x \mid x \in \mathbb{R}\}$, and the range is all real numbers greater than 0, $\{y \mid y \geq 0\}$.

13. What are the intercepts of the function that models the data that Noah collected?
The y-intercept is (0, 28) and the x-intercept is (24, 0).

14. What is the maximum or minimum value of the function that models the data that Noah collected?
The minimum value is the point (24, 0).

15. Graph the functions $y = |x|$, $y = 2|x|$, $y = 4|x|$, $y = 0.5|x|$, and $y = -0.5|x|$ on the same screen of your graphing calculator. Does the parameter a affect the absolute value parent function similarly as other functions? Explain how you know.
See margin.

> The point of the *V* in the graph of an absolute value function is called the **vertex**. As in a quadratic function, the vertex is the point where the rate of change of the function changes from decreasing to increasing if the graph is concave up or from increasing to decreasing if the graph is concave down.

16. Graph the functions $y = |x|$, $y = |3x|$, $y = |4x|$, $y = |0.5x|$, and $y = |-0.5x|$ on the same screen of your graphing calculator. Does the parameter b affect the absolute value parent function similarly as other functions? Explain how you know.
See margin.

17. Graph the functions $y = |x|$, $y = |x - 3|$, $y = |x - 5|$, $y = |x + 2|$, and $y = |x + 6|$ on the same screen of your graphing calculator. Does the parameter c affect the absolute value parent function similarly as other functions? Explain how you know.
See margin.

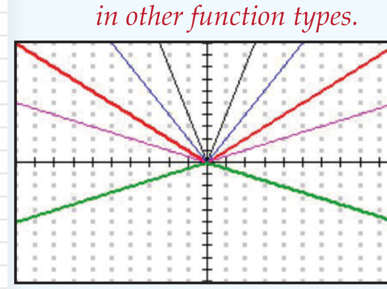
18. Graph the functions $y = |x|$, $y = |x| + 3$, $y = |x| + 5$, $y = |x| - 4$, and $y = |x| - 3$ on the same screen of your graphing calculator. Does the parameter d affect the absolute value parent function similarly as other functions? Explain how you know.
See margin.

SUPPORTING ENGLISH LANGUAGE LEARNERS

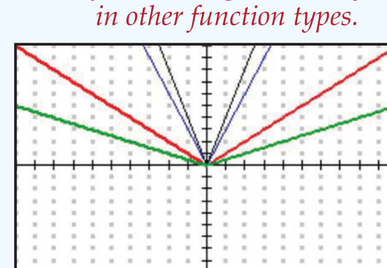
Students have seen the term absolute value in middle school. They have also worked in middle school and Algebra 1 extensively with linear functions. In this lesson, the accessible terms absolute value, linear, function, domain, and range come together for students to learn new and essential language relating to absolute value functions. Making those connections explicit supports students who are learning the English language.

11. *In the function $f(x) = \frac{7}{6}|x - 24|$, $a = \frac{7}{6}$ and $c = 24$. This function is the parent function dilated vertically by a factor of $= \frac{7}{6}$ and translated horizontally 24 units to the right.*

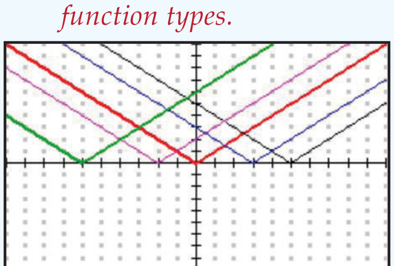
15. *Yes. $|a| > 1$ generates a vertical dilation (stretch), when $0 < |a| < 1$ generates a vertical dilation (compression), and when $a < 0$ generates a vertical reflection across the x-axis. These are the same transformations generated by a in other function types.*



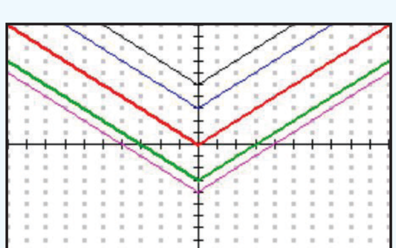
16. *Yes. $|b| > 1$ generates a horizontal dilation (stretch) by a factor of $\frac{1}{|b|}$, when $0 < |b| < 1$ generates a horizontal dilation (compression) by a factor of $\frac{1}{|b|}$, and when $b < 0$ generates a horizontal reflection across the y-axis. These are the same transformations generated by b in other function types.*



17. *Yes. $c > 0$ generates a horizontal translation $\left|\frac{c}{b}\right|$ units to the right and $c < 0$ generates a horizontal translation $\left|\frac{c}{b}\right|$ units to the left. These are the same transformations generated by c in other function types.*



18. *Yes. $d > 0$ generates a vertical translation $|d|$ units up and $d < 0$ generates a vertical translation $|d|$ units down. These are the same transformations generated by d in other function types.*



REFLECT ANSWERS:

The domain for all absolute value functions is all real numbers, so a parameter change does not affect the domain. The range is affected by d, which vertically shifts the vertex of the absolute value graph, which is the maximum or minimum value of the absolute value function and a, which reflects the function over the x-axis.

The x-coordinate of the vertex is affected by the parameters b and c, which influence horizontal dilations and horizontal translations. The y-coordinate of the vertex is affected by the parameter d, which influences vertical translations. The parameter a does not affect the y-intercept because vertical dilations are performed with respect to the line y = d and since the vertex lies on this line, its coordinates will not change when a changes.

The y-intercept is affected by the parameters a, c, and d. The parameters c and d influence horizontal and vertical translations that move the vertex of the graph. The parameter a influences vertical dilations and reflections which stretch, compress, or reflect the y-intercept. The parameter b does not affect the y-intercept because horizontal dilations are performed with respect to the y-axis, which has a value of x = 0.

REFLECT

- **Which parameters affect the domain and range of an absolute value function? Why do you think that is the case?**
  See margin.
- **Which parameters would affect the vertex of an absolute value function? Why do you think that is the case?**
  See margin.
- **Which parameters affect the *y*-intercept of an absolute value function? Why do you think that is the case?**
  See margin.

EXPLAIN

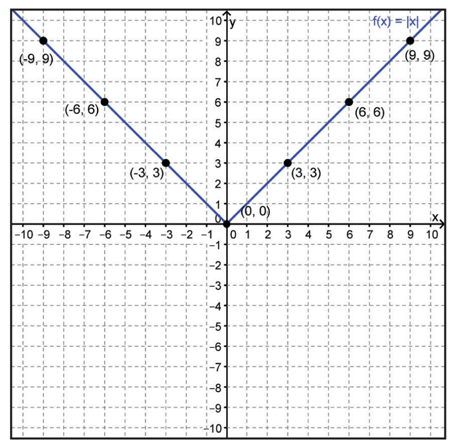
An absolute value function is a relationship between an independent variable, usually x, and a dependent variable, usually y or $f(x)$, that generates a graph in the shape of a V. Absolute value functions of the form $y = a|bx - c| + d$ have two linear branches connected at the vertex of the V that are reflections of each other.

Watch Explain and You Try It Videos

or click here

$f(x) = |x|$

x	$f(x)$
-9	9
-6	6
-3	5
0	0
3	5
6	6
9	9



Applying transformations to the absolute value parent function generates the full family of absolute value functions. Transformations are applied using parameters that are multiplied or added to the independent variable in the functional relationship.

For an absolute value function, the general form is $f(x) = a|bx - c| + d$, where a, b, c, and d are real numbers.

CHANGES IN *a*

The parameter a influences the vertical dilation of the graph.

- If $|a| > 1$, then the range values (y-coordinates) of the original function are multiplied by a factor of $|a|$ in order to vertically stretch the graph.
- If $0 < |a| < 1$, then the range values (y-coordinates) of the original function are multiplied by a factor of $|a|$ in order to vertically compress the graph.
- If $a < 0$, then the graph will be reflected across the x-axis.

INSTRUCTIONAL HINT

Have students continue their graphic organizers from sections 2.1-2.3.

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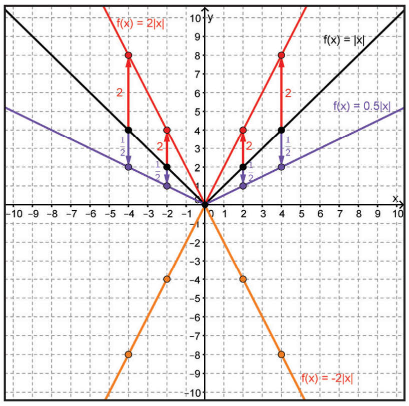


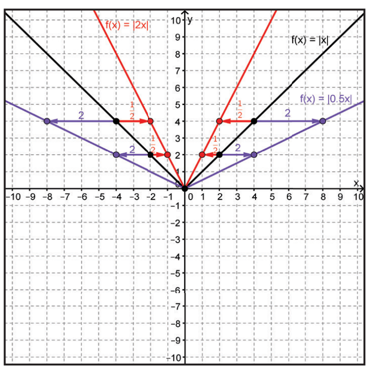
x	$f(x) = \lvert x \rvert$	$f(x) = 2\lvert x \rvert$	$f(x) = 0.5\lvert x \rvert$	$f(x) = -2\lvert x \rvert$
-4	4	8	2	-8
-2	2	4	1	-4
0	0	0	0	0
2	2	4	1	-4
4	4	8	2	-8

× 2 × 0.5 × (-1)

CHANGES IN b

The parameter b influences the horizontal stretch or compression of the graph.

- If $|b| > 1$, then the domain values (x-coordinates) of the original function are multiplied by a factor of $\frac{1}{|b|}$, which will be a multiplier that is less than 1, in order to horizontally compress the graph.
- If $0 < |b| < 1$, then the domain values (x-coordinates) of the original function are multiplied by a factor of $\frac{1}{|b|}$, which will be a multiplier that is greater than 1, in order to horizontally stretch the graph.



x	$f(x) = \lvert x \rvert$	x	$f(x) = \lvert 2x \rvert$	x	$f(x) = x$	x	$f(x) = \lvert 0.5x \rvert$
-4	4	-2	4	-4	4	-8	4
-2	2	-1	2	-2	2	-4	2
0	0	0	0	0	0	0	0
2	2	1	2	2	2	4	2
4	4	2	4	4	4	8	4

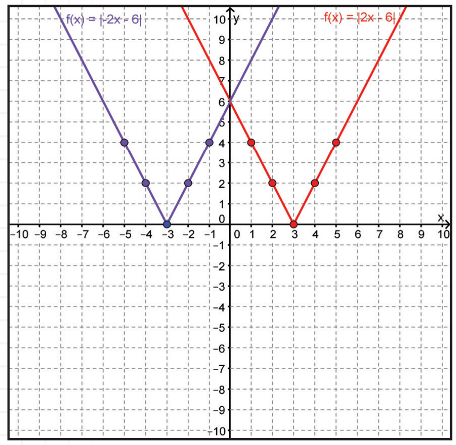
Domain (x) values are multiplied by $\frac{1}{2}$ in order to generate the same range (y) value. This multiplication results in a horizontal compression of the graph.

Domain (x) values are multiplied by 2 in order to generate the same range (y) value. This multiplication results in a horizontal stretch of the graph.

The parameter b also affects the orientation of the graph. If $b < 0$, then all of the x-values will change signs and the graph will be reflected across the y-axis.

QUESTIONING STRATEGY

With linear, quadratic, and cubic functions, the influence of b on the functions is always written as $\frac{1}{|b|}$. Why is b not always written as the absolute value with absolute value functions?



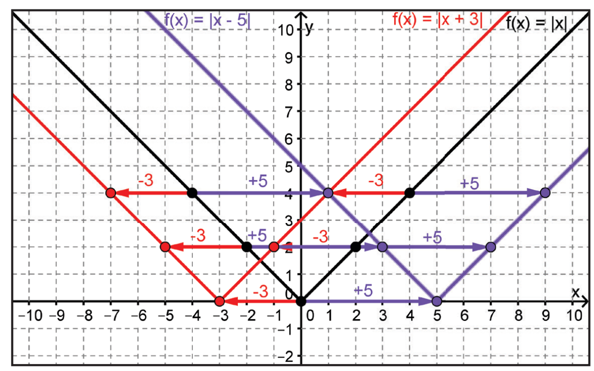
x	$f(x) = \lvert 2x - 6 \rvert$
1	4
2	2
3	0
4	2
5	4

x	$f(x) = \lvert -2x - 6 \rvert$
-1	4
-2	2
-3	0
-4	2
-5	4

Domain (x) values are multiplied by −1, which changes the sign but not the magnitude of each x-value. This change results in a horizontal reflection of the graph across the y-axis.

CHANGES IN c

The parameter c influences the horizontal translation of the graph. Notice that in the general form, $f(x) = a\lvert bx - c \rvert + d$, the sign in front of c is negative. That means that when reading the value of c from the equation, you should read the opposite sign from what is given in the equation. For example, $f(x) = \lvert 3x - 7 \rvert + 1$ has $c = 7$ and $f(x) = \lvert 3x + 7 \rvert + 1$ has $c = -7$.



- If $c > 0$, then the graph will translate $\lvert \frac{c}{b} \rvert$ units to the right.
- If $c < 0$, then the graph will translate $\lvert \frac{c}{b} \rvert$ units to the left.

x	$f(x) = \lvert x \rvert$
-4	4
-2	2
0	0
2	2
4	4

x	$f(x) = \lvert x - 5 \rvert$
1	4
3	2
5	0
7	2
9	4

Domain (x) values are increased by 5 in order to generate the same range (y) value. This addition results in a horizontal translation of the graph to the right.

x	$f(x) = \lvert x \rvert$
-4	4
-2	2
0	0
2	2
4	4

x	$f(x) = \lvert x + 3 \rvert$
-7	4
-5	2
-3	0
-1	2
1	4

Domain (x) values are decreased by 3 in order to generate the same range (y) value. This subtraction (addition with a negative number) results in a horizontal translation of the graph to the left.

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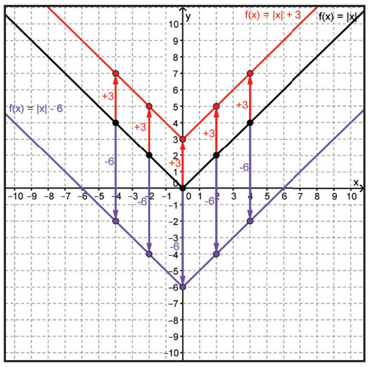
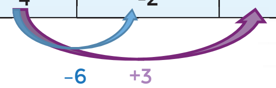
CHANGES IN *d*

The parameter *d* influences the horizontal translation of the graph.

- If $d > 0$, then the graph will translate $|d|$ units up.
- If $d < 0$, then the graph will translate $|d|$ units down.



x	$f(x) = \|x\|$	$f(x) = \|x\| - 6$	$f(x) = \|x\| + 3$
-4	4	-2	7
-2	2	-4	5
0	0	-6	3
2	2	-4	5
4	4	-2	7



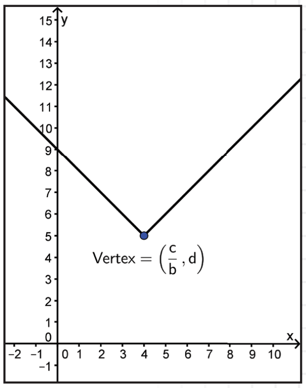
Each of these parameters also has an effect on key attributes of a quadratic function.

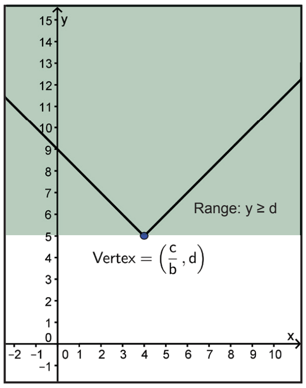
VERTEX

The **vertex** of the absolute value graph is a maximum or minimum value. If the graph opens upward, then the *y*-coordinate of the vertex is the minimum value for the function. If the graph opens downward, then the *y*-coordinate of the vertex is the maximum value for the function.

As with a quadratic function, three parameters influence the location of the vertex: *b*, *c*, and *d*. The coordinates of the vertex of an absolute value graph can be found using the same formulas as a quadratic function.

$$\text{Vertex: } \left(\frac{c}{b}, d\right)$$





DOMAIN AND RANGE

All real numbers have an absolute value, so there are no domain restrictions on absolute value functions. The domain is always all real numbers, or $\{x \mid x \in \mathbb{R}\}$.

The range of an absolute value function, however, does have restrictions. Taking the absolute value of a number generates a positive number. As with quadratic functions, the absolute value action always generates a positive number.

The parameter affecting the range is *d*. If the graph opens upward, then the value of *d* sets the *y*-coordinate of the vertex at a minimum value, and the range is $y \geq d$.

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ADDITIONAL EXAMPLES

Recall prior learning by identifying whether the following tables represent a linear, quadratic, or absolute value function.

1.

x	y
-3	-15
-2	-10
-1	0
0	15
1	35
2	60
3	90

Quadratic

2.

x	y
-3	-7
-2	-10
-1	-13
0	-16
1	-13
2	-10
3	-7

Absolute Value

3.

x	y
-3	22.5
-2	24
-1	25.5
0	27
1	28.5
2	30
3	31.5

Linear

If the graph opens downward, then the value of d sets the y-coordinate of the vertex at a maximum value, and the range is $y \leq d$.

y-INTERCEPT

The y-intercept of any function is the point $(0, y)$, or the point where the graph intersects the y-axis. For an absolute value function, you can calculate the y-intercept by substituting $x = 0$ into the general form, $y = a|bx - c| + d$.

$$y = a|b(0) - c| + d$$
$$y = a|0 - c| + d$$
$$y = a|-c| + d$$
$$y = a|c| + d$$

The parameters a, c, and d each influence the y-intercept of an absolute value function. The parameter d vertically shifts the entire graph, including the y-intercept, and the parameter c horizontally shifts the entire graph, including the y-intercept. The parameter a vertically dilates and/or reflects the entire graph, including the y-intercept. Each of these transformations affects where the graph intersects the y-axis.

KEY ATTRIBUTES OF ABSOLUTE VALUE FUNCTIONS

An absolute value function has several important key attributes:

- The domain of an absolute value function is all real numbers.
- The range of an absolute value function is $\{y \mid y \geq d\}$ for $a > 0$ and $\{y \mid y \leq d\}$ for $a < 0$.
- The vertex of an absolute value function is $\left(\frac{c}{b}, d\right)$. If $a > 0$ the y-coordinate of the vertex is a minimum value. If $a < 0$, the y-coordinate of the vertex is a maximum value.
- An absolute value function has as many as two x-intercepts that are the same as the zeros of the function. The x-intercepts are located at $\left(\frac{c \pm \frac{d}{a}}{b}, 0\right)$.
- An absolute value function has one y-intercept at $(0, a|c| + d)$.

EXAMPLE 1

What transformations of the absolute value parent function, $f(x) = |x|$, will result in the graph of the absolute value function $g(x) = -\frac{1}{2}|2x + 1| - 3$?

STEP 1 Rewrite the equation of $g(x)$ in general form to determine the values of the parameters a, b, c and d.

$g(x) = -\frac{1}{2}|2x + 1| - 3$

$g(x) = -\frac{1}{2}|2x - (-1)| + (-3)$

Therefore, $a = -\frac{1}{2}$, $b = 2$, $c = -1$, and $d = -3$.

STEP 2 **Use the values of the parameters to describe the transformations of the absolute value parent function $f(x)$ that are necessary to produce $g(x)$.**

$a = -\frac{1}{2}$, so $0 < |a| < 1$. The range values (y-coordinates) of the absolute value parent function are multiplied by a factor of $\frac{1}{2}$ in order to vertically compress the graph of the line. Additionally, since a is negative, the graph is reflected across the x-axis.

$b = 2$, so $|b| > 1$. The domain values (x-coordinates) of the original function are multiplied by a factor of $\frac{1}{2}$ in order to horizontally compress the graph.

$c = -1$, so $c < 0$. The graph of the quadratic parent function will translate $|-\frac{1}{2}| = \frac{1}{2}$ unit to the left.

$d = -3$, so $d < 0$. The graph of the quadratic parent function will translate $|-3| = 3$ units down.

The graph of g(x) is produced by transforming the absolute value parent function f(x) by vertically compressing its graph by a factor of one half, horizontally compressing its graph by a factor of one half, reflecting its graph over the x-axis, and translating its graph one half of a unit to the left and three units down.

YOU TRY IT! #1

What transformations of the absolute value parent function, $y = |x|$, will result in the graph of the absolute value function $y = 3|\frac{1}{2}x - 5|$?

See margin.

ADDITIONAL EXAMPLES

What transformations of the absolute value parent function, $f(x) = |x|$, will result in the graphs of the absolute value functions below?

1. $h(x) = -10|x - 4| + 12$

The graph of h(x) is produced by transforming the absolute value parent function f(x) by vertically stretching its graph by a factor of ten, reflecting its graph across the x-axis, and translating its graph four units to the right and twelve units up.

2. $v(x) = \frac{1}{3}|-4x - 5| - 3$

The graph of v(x) is produced by transforming the absolute value parent function f(x) by vertically compressing its graph by a factor of one third, horizontally compressing its graph by a factor of one fourth, reflecting its graph across the y-axis, and translating its graph left five fourths units and down three units.

QUESTIONING STRATEGY

Why did **ADDITIONAL EXAMPLE #2** translate left instead of right when the value of c was greater than 1?

YOU TRY IT! #1 ANSWER:

The graph of $y = 3|\frac{1}{2}x - 5|$ is produced by transforming the absolute value parent function by vertically stretching its graph by a factor of three, horizontally stretching its graph by a factor of two, and translating its graph ten units to the right.

ADDITIONAL EXAMPLES

Identify the key attributes of the functions below including domain, range, vertex, x-intercept(s), and y-intercept. Write the domain and range as intervals and in set builder notation. Determine whether the vertex is a maximum or a minimum value of the function.

1. $y = -\frac{3}{2}|4x - 16| + 4$

The domain is (-∞, ∞) or {x | ∈ ℝ}. The range is (-∞, 4] or {y | y ≤ 4}. The vertex of the V-shaped graph is (4, 4), and it is the maximum function value. This absolute value function has x-intercepts at (3.33, 0) and (4.67, 0). The y-intercept is (0, -20).

2. $y = |-8x + 4.25|$

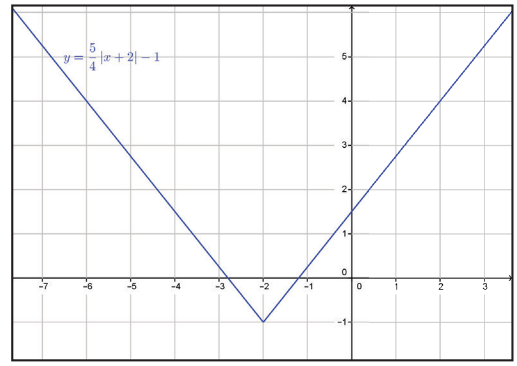
The domain is (-∞, ∞) or {x | ∈ ℝ}. The range is [0, ∞) or {y | y ≥ 0}. The vertex of the V-shaped graph is (-0.53, 0), and it is the minimum function value. This absolute value function has one x-intercept at (-0.53, 0). The y-intercept is (0, 4.25).

3. $y = 4|-\frac{1}{2}x + 3| - 2.5$

The domain is (-∞, ∞) or {x | ∈ ℝ}. The range is [-2.5, ∞) or {y | y ≥ -2.5}. The vertex of the V-shaped graph is (-6, -2.5), and it is the minimum function value. This absolute value function has x-intercepts at (7.25, 0) and (4.75, 0). The y-intercept is (0, 9.5).

EXAMPLE 2

Identify the key attributes of $y = \frac{5}{4}|x + 2| - 1$, including domain, range, vertex, x-intercept(s), and y-intercept. Write the domain and range as intervals and in set builder notation. Determine whether the vertex is a maximum or a minimum value of the function.



STEP 1 **Determine the domain and range of $y = \frac{5}{4}|x + 2| - 1$.**

From the graph, the domain contains all real values of x. As an interval, this is written as $(-\infty, \infty)$. The same information about the domain written in set builder notation is $\{x \mid x \in \mathbb{R}\}$.

Using the equation, the value of a indicates that the V-shaped graph of the function opens upward. This is confirmed by the provided graph. The only other parameter that affects the range of the absolute function is d. Since $d = -1$, the range of this absolute function contains all real values of y that are greater than or equal to negative one. As an interval, this is written as $[-1, \infty)$. The same information about the range written in set builder notation is $\{y \mid y \geq -1\}$.

STEP 2 **Determine the vertex of the V-shaped graph of the absolute value function.**

The vertex of an absolute value function written in the general form $y = a|bx - c| + d$ is $\left(\frac{c}{b}, d\right)$.

$\frac{c}{b} = \frac{-2}{1} = -2$ and $d = -1$.

The vertex of the absolute value function is $(-2, -1)$. Since the value of a is positive, the vertex is a minimum. This is confirmed by the graph.

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STEP 3 **Determine the x-intercept(s) of the absolute value function $y = \frac{5}{4}|x + 2| - 1$.**

If they exist, the x-intercepts are located at $\left(\frac{c \pm \frac{d}{a}}{b}, 0\right)$.

$$\frac{-2 \pm \frac{-1}{\frac{5}{4}}}{1} = \frac{-2 \pm \frac{-4}{5}}{1} = -2 \pm -\frac{4}{5}$$

$$-2 + -\frac{4}{5} = -\frac{14}{5} = -2.8$$

$$-2 - -\frac{4}{5} = -2 + \frac{4}{5} = -1.2$$

The x-intercepts of the absolute value graph are (–2.8, 0) and (–1.2, 0). The graph confirms this since the V-shaped graph intersects the x-axis between –3 and –2 as well as between –2 and –1.

STEP 4 **Determine the y-intercept of $y = \frac{5}{4}|x + 2| - 1$.**

y-intercepts of functions occur where the domain or input value $x = 0$.

$$y = \frac{5}{4}|x + 2| - 1$$

$$y = \frac{5}{4}|0 + 2| - 1 = \frac{5}{4}|2| - 1 = \frac{5}{4}(2) - 1 = \frac{10}{4} - 1 = \frac{6}{4} = 1.5.$$

The y-intercept of the quadratic function is (0, 1.5). The graph confirms this since the line intersects the y-axis at a point between (0, 1) and (0, 2).

The domain of $y = \frac{5}{4}|x + 2| - 1$ is $(-\infty, \infty)$ or $\{x \mid x \in \mathbb{R}\}$. The range is $[-1, \infty)$ which may also be written as $\{y \mid y \geq -1\}$. The vertex of the V-shaped graph is (–2, –1), and it is the minimum function value. This absolute value function has x-intercepts at (–2.8, 0) and (–1.2, 0). The y-intercept of $y = \frac{5}{4}|x + 2| - 1$ is (0, 1.5).

YOU TRY IT! #2

Identify the key attributes of $f(x) = -|4x| - 2$, including its domain, range, vertex, x-intercept(s), and y-intercepts. Write the domain and range of $f(x)$ as inequalities and in set builder notation. Determine whether the vertex is a maximum or minimum value.

YOU TRY IT! #2 ANSWER:

The domain of $f(x)$ is $-\infty < x < \infty$, which can also be written as $\{x \mid x \in \mathbb{R}\}$. The range of $f(x)$ is $-\infty < f(x) \leq -2$ or $\{f(x) \mid f(x) \leq -2\}$. The vertex of the V-shaped graph is $(0, -2)$, and it is a maximum function value. The function has no x-intercepts since its range does not include zero. The y-intercept is $(0, -2)$, the same as the vertex of the graph.

x	$f(x)$
-3	-14
-2	-10
-1	-6
0	-2
1	-6
2	-10
3	-14

See margin.

EXAMPLE 3

Identify and compare the x- and y-intercept(s) of $f(x) = -2(x + 3)$, $g(x) = -2(x + 3)^2$, and $h(x) = -2|x + 3|$.

STEP 1 **Determine the x-intercept(s) of $f(x)$.**

Since $f(x)$ is a linear function, $f(x)$ has one x-intercept at $\left(\frac{ac - d}{ab}, 0\right)$.

$$\frac{ac - d}{ab} = \frac{(-2)(-3) - 0}{(-2)(1)} = \frac{6}{-2} = -3$$

The x-intercept of $f(x)$ is $(-3, 0)$.

STEP 2 **Determine the x-intercept(s) of $g(x)$.**

Since $g(x)$ is a quadratic function, the x-intercepts are located at $\left(\frac{c \pm \sqrt{\frac{-d}{a}}}{b}, 0\right)$.

$$\frac{c \pm \sqrt{\frac{-d}{a}}}{b} = \frac{-3 \pm \sqrt{\frac{-0}{-2}}}{1} = \frac{-3 \pm \sqrt{0}}{1} = \frac{-3 \pm 0}{1} = -3$$

The x-intercept of $g(x)$ is $(-3, 0)$.

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STEP 3 **Determine the x-intercept(s) of $h(x)$.**

If they exist, the x-intercepts are located at $\left(\frac{c \pm \frac{d}{a}}{b}, 0\right)$.

$\frac{-3 \pm \frac{0}{-2}}{1} = \frac{-3 \pm 0}{1} = -3$

The x-intercept of the $h(x)$ is (–3, 0).

STEP 4 **Compare the x-intercepts of $f(x)$, $g(x)$ and $h(x)$.**

Since $f(x)$ is a linear function, it has only one x-intercept. The quadratic function $g(x)$ also has only one x-intercept. The absolute value function $h(x)$ has just one x-intercept. All three functions have the same x-intercept, (–3, 0). This is due to the parameter c having the same value and the parameter d being zero for all three functions.

STEP 5 **Determine the y-intercepts of $f(x)$, $g(x)$ and $h(x)$.**

y-intercepts of functions occur where the domain or input value $x = 0$.

$f(x) = -2(x + 3)$

$f(0) = -2(0 + 3) = -2(3) = -6$

The y-intercept of $f(x)$ is (0, –6).

$g(x) = -2(x + 3)^2$

$g(0) = -2(0 + 3)^2 = -2(3)^2 = -2(9) = -18$

The y-intercept of $g(x)$ is (0, –18).

$h(x) = -2|x + 3|$

$h(0) = -2|0 + 3| = -2|3| = -2(3) = -6$

The y-intercept of $h(x)$ is (0, –6).

STEP 6 **Compare the y-intercepts of $f(x)$, $g(x)$ and $h(x)$.**

The y-intercepts of the linear function $f(x)$ and the absolute value function $h(x)$ are the same, (0, –6). The y-intercept of the quadratic function $g(x)$ is (0, –18). The difference in the y-intercept of the quadratic function is due to the squaring that is involved in quadratic functions that does not exist in either linear or absolute value functions.

ADDITIONAL EXAMPLES

Identify and compare the x- and y-intercepts of the functions below.

1.

$f(x) = 3|-2x + 1| - 12$

$g(x) = 3(-2x + 1) - 12$

$h(x) = 3(-2x + 1)^2 - 12$

The x-intercepts of f(x) are ($\frac{5}{2}$, 0) and (-$\frac{3}{2}$, 0). The y-intercept is (0, -9).

The x-intercept of g(x) is (-$\frac{3}{2}$, 0). They y-intercept is (0, -9).

The x-intercepts for h(x) are (-$\frac{1}{2}$, 0) and ($\frac{3}{2}$, 0). The y-intercept is (0, -9).

2.

$f(x) = -\frac{1}{5}|10x - 5| + 20$

$g(x) = -\frac{1}{5}(10x - 5) + 20$

$h(x) = -\frac{1}{5}(10x - 5)^2 + 20$

The x-intercepts of f(x) are (-$\frac{19}{2}$, 0) and ($\frac{21}{2}$, 0). The y-intercept is (0, 19).

The x-intercept of g(x) is ($\frac{21}{2}$, 0). They y-intercept is (0, 21).

The x-intercepts for h(x) are ($\frac{3}{2}$, 0) and ($\frac{1}{2}$, 0). The y-intercept is (0, 15).

Since f(x) is a linear function, it has only one x-intercept. The quadratic function g(x) also has only one x-intercept since its vertex is on the x-axis rather than above or below it. The absolute value function h(x) has just one x-intercept since its vertex is also on the x-axis rather than above or below it. All three functions have the same x-intercept, (–3, 0). This is due to the parameter c having the same value and the parameter d being zero for all three functions.

The y-intercepts of the linear function f(x) and the absolute value function h(x) are the same, (0, –6). The y-intercept of the quadratic function g(x) is (0, –18). The difference in the y-intercept of the quadratic function is due to the squaring that is involved in quadratic functions that does not exist in either linear or absolute value functions.

YOU TRY IT! #3 ANSWER:

The y-intercept of f(x) is (0, –11.5). The y-intercept of g(x) is (0, 3.5). The y-intercept of h(x) is (0, –1.5). Even though the parameters a, c, and d are the same for all three functions, none of the y-intercepts are the same. If the value of c had been negative, then the y-intercepts of g(x) and h(x) would have been the same. It is important to note that the differences in the y-intercepts are not due to the parameter b, which was eliminated each time because it was multiplied by zero to determine the y-intercept.

YOU TRY IT! #3

Identify and compare the y-intercept(s) of $f(x) = -\frac{1}{2}(3x - 5)^2 + 1$, $g(x) = -\frac{1}{2}(3x - 5) + 1$, and $h(x) = -\frac{1}{2}|3x - 5| + 1$.

See margin.

EXAMPLE 4

Identify and compare the domains and ranges of $f(x) = (2x - 7)^2 + 3$, $g(x) = (2x - 7) + 3$, and $h(x) = |2x - 7| + 3$. Write the domain and range of each function as inequalities, as intervals, and in set builder notation.

STEP 1 **Determine the domain and range of *f(x)*.**

Since $f(x)$ is a quadratic function, the domain contains all real values of x. As an interval, this is written as $(-\infty, \infty)$. The same information about the domain written in set builder notation is $\{x \mid x \in \mathbb{R}\}$. Written as an inequality, $-\infty < x < \infty$.

Using the equation, the value of a indicates that the parabola opens upward. The only other parameter that affects the range of a quadratic function is d. Since $d = 3$, the range of this quadratic function contains all real values of $f(x)$ that are greater than or equal to three. As an interval, this is written as $[3, \infty)$. The same information about the range written in set builder notation is $\{f(x) \mid f(x) \geq 3\}$. As an inequality, this is written $3 \leq f(x) < \infty$.

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STEP 2 **Determine the domain and range of *g(x)*.**

Since $g(x)$ is a linear function, its domain contains all real values of x. As an interval, this is written as $(-\infty, \infty)$. The same information about the domain written in set builder notation is $\{x \mid x \in \mathbb{R}\}$. Written as an inequality, $-\infty < x < \infty$.

Since $g(x)$ is a linear function, its range contains all real values of $g(x)$. As an interval, this is written as $(-\infty, \infty)$. The same information about the range written in set builder notation is $\{g(x) \mid g(x) \in \mathbb{R}\}$. Written as an inequality, $-\infty < g(x) < \infty$.

STEP 3 **Determine the domain and range of *h(x)*.**

Since $h(x)$ is an absolute value function, the domain contains all real values of x. As an interval, this is written as $(-\infty, \infty)$. The same information about the domain written in set builder notation is $\{x \mid x \in \mathbb{R}\}$. Written as an inequality, $-\infty < x < \infty$.

Using the equation, the value of a, which is positive one, indicates that the V-shaped graph opens upward. The only other parameter that affects the range of an absolute value function is d. Since $d = 3$, the range of this absolute value function contains all real values of $h(x)$ that are greater than or equal to three. As an interval, this is written as $[3, \infty)$. The same information about the range written in set builder notation is $\{h(x) \mid h(x) \geq 3\}$. As an inequality, this is written $3 \leq h(x) < \infty$.

STEP 4 **Compare the domains and ranges of *f(x)*, *g(x)* and *h(x)*.**

The domains of all three of these functions are the same because both $f(x)$ and $g(x)$ are polynomials, one of degree one and the other of degree two, and $h(x)$ is an absolute value function. The range of $g(x)$ contains all real numbers since $g(x)$ is a linear function. The ranges of the quadratic function $f(x)$ and the absolute value function $h(x)$ do not contain all real numbers but are restricted to only those real numbers greater than or equal to three.

ADDITIONAL EXAMPLE

Identify and compare the domains and ranges of the functions below. Write the domain and range of each function as inequalities, as intervals, and in set builder notation.

$f(x) = \frac{1}{2}(-4x + 16) - 5$

$g(x) = \frac{1}{2}(-4x + 16)^2 - 5$

$h(x) = \frac{1}{2}|-4x + 16| - 5$

The domains of all of these functions contain all real numbers.

- *Inequality:* $-\infty < x < \infty$
- *Interval:* $(-\infty, \infty)$
- *Set Builder:* $\{x \mid x \in \mathbb{R}\}$

The range of f(x) contains all real numbers.

- *Inequality:* $-\infty < x < \infty$
- *Interval:* $(-\infty, \infty)$
- *Set Builder:* $\{x \mid x \in \mathbb{R}\}$

The ranges of g(x) and h(x) do not contain all real numbers but are restricted to only those real numbers greater than or equal to -5.

- *Inequality:* $-5 \leq g(x) < \infty$
- *Interval:* $[-5, \infty)$
- *Set Builder:* $\{g(x) | g(x) \geq -5\}$
- *Inequality:* $-5 \leq h(x) < \infty$
- *Interval:* $[-5, \infty)$
- *Set Builder:* $\{h(x) | h(x) \geq -5\}$

	NOTATION	$f(x) = (2x-7)^2 + 3$	$g(x) = (2x-7) + 3$	$h(x) = \lvert 2x-7 \rvert + 3$
DOMAIN	INEQUALITY	$-\infty < x < \infty$		
	INTERVAL	$(-\infty, \infty)$		
	SET BUILDER	$\{x \mid x \in \mathbb{R}\}$		
RANGE	INEQUALITY	$3 \leq f(x) \leq \infty$	$-\infty < g(x) < \infty$	$3 \leq h(x) \leq \infty$
	INTERVAL	$[3, \infty)$	$(-\infty, \infty)$	$[3, \infty)$
	SET BUILDER	$\{f(x) \mid f(x) \geq 3\}$	$\{g(x) \mid g(x) \in \mathbb{R}\}$	$\{h(x) \mid h(x) \geq 3\}$

The domains of all three of these functions are the same because both f(x) and g(x) are polynomials, one of degree one and the other of degree two, and h(x) is an absolute value function. The range of g(x) contains all real numbers since g(x) is a linear function. The ranges of the quadratic function f(x) and the absolute value function h(x) do not contain all real numbers but are restricted to only those real numbers greater than or equal to the value of its parameter d. The functions f(x) and h(x) have the same range because their parameters a have the same sign and d are identical.

YOU TRY IT! #4

Identify and compare the domains and ranges of $f(x) = -3(2x) - 0.5$, $g(x) = -3(2x)^2 - 0.5$ and $h(x) = -3\lvert 2x \rvert - 0.5$. Write the domain and range of each function as inequalities, as intervals, and in set builder notation.

See margin.

PRACTICE/HOMEWORK

For questions 1 – 4, use finite differences to determine if the data sets shown in the tables below represent linear, quadratic, or absolute value functions.

1.

x	y
-3	23
-2	13
-1	7
0	5
1	7
2	13
3	23

Quadratic

2.

x	y
-3	11
-2	9
-1	7
0	5
1	7
2	9
3	11

Absolute Value

3.

x	y
-3	-1
-2	1
-1	3
0	5
1	7
2	9
3	11

Linear

4.

x	y
-3	7
-2	5
-1	3
0	5
1	7
2	9
3	11

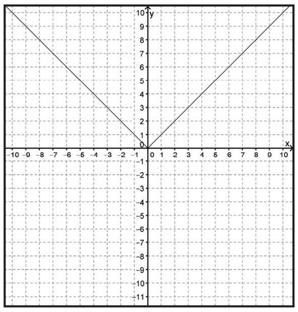
Absolute Value

YOU TRY IT! #4 ANSWER:

The domains of all three of these functions are the same because both f(x) and g(x) are polynomials, one of degree one and the other of degree two, and h(x) is an absolute value function. The range of f(x) contains all real numbers since f(x) is a linear function. The ranges of the quadratic function g(x) and the absolute value function h(x) do not contain all real numbers but are restricted to only those real numbers less than or equal to the value of their parameters d. The reason the ranges of g(x) and h(x) contain values less than or equal to negative five tenths rather than greater or equal to is that the value of their parameters a are negative.

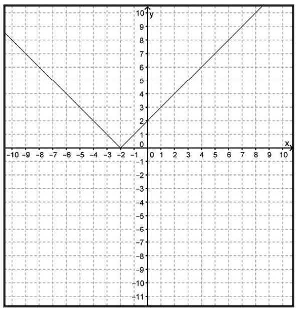
	NOTATION	$f(x) = -3(2x) - 0.5$	$g(x) = -3(2x)^2 - 0.5$	$h(x) = -3\lvert 2x \rvert - 0.5$
DOMAIN	INEQUALITY	$-\infty < x < \infty$		
	INTERVAL	$(-\infty, \infty)$		
	SET BUILDER	$\{x \mid x \in \mathbb{R}\}$		
RANGE	INEQUALITY	$-\infty \leq f(x) < \infty$	$-\infty < g(x) \leq -0.5$	$-\infty < h(x) \leq -0.5$
	INTERVAL	$(-\infty, \infty)$	$(-\infty, -0.5]$	$(-\infty, -0.5]$
	SET BUILDER	$\{f(x) \mid f(x) \in \mathbb{R}\}$	$\{g(x) \mid g(x) \leq -0.5\}$	$\{h(x) \mid h(x) \leq -0.5\}$

The graph of the parent function $f(x) = |x|$ is shown below.



For questions 5 – 12 look at the given graph and determine the correct function represented on the graph based on transformations made to the parent function.

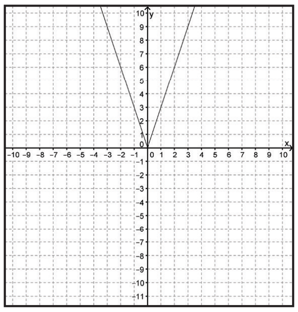
5.



A. $y = |x + 2|$ or **B.** $y = |x - 2|$

A

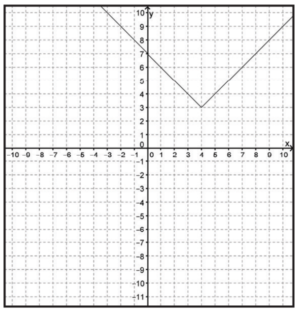
6.



A. $y = -|3x|$ or **B.** $y = |-3x|$

B

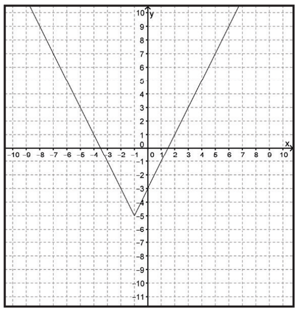
7.



A. $y = |x + 4| + 3$ or **B.** $y = |x - 4| + 3$

B

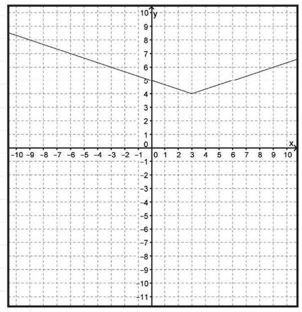
8.



A. $y = \frac{1}{2}|x + 1| - 5$ or **B.** $y = 2|x + 1| - 5$

B

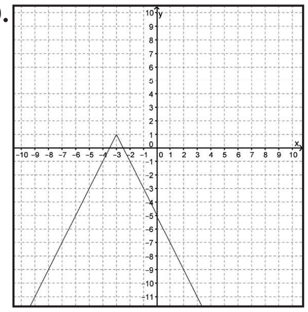
9.



A. $y = \frac{1}{3}|x - 3| + 4$ or **B.** $y = 3|x - 3| + 4$

A

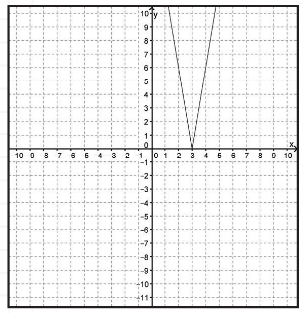
10.



A. $y = |-2x - 6| + 1$ or **B.** $y = -|-2x - 6| + 1$

B

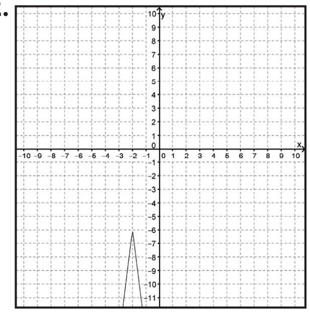
11.



A. $y = 3|2x - 6|$ or **B.** $y = 2|3x - 6|$

A

12.



A. $y = -4|2x + 4| - 6$ or **B.** $y = -2|4x - 8| - 6$

A

For questions 13 – 14, the data sets shown in the tables represent absolute value functions.

A. Write a pair of linear functions that models the data.

B. Write the pair of functions as one absolute value function.

C. Write the absolute function in the form $f(x) = a|x - c|$

13.

x	y
-7	3
-6	2
-5	1
-4	0
-3	1
-2	2
-1	3

See margin.

14.

x	y
0	-6
1	-4
2	-2
3	0
4	-2
5	-4
6	-6

See margin.

13. *A.* $y = -x - 4$ $\quad y = x + 4$

B. $f(x) = |x + 4|$

C. $f(x) = \frac{1}{2}|2x + 8|$

14. *A.* $y = 2x - 6$ $\quad y = -2x + 6$

B. $f(x) = -|2x - 6|$

C. $f(x) = -2(x - 3)$

For questions 15 – 16, identify the vertex of the V-shaped graph of each absolute value function.

15. $y = -2|3x - 6| + 1$

(2, 1)

16. $y = \frac{1}{3}|-2x + 7| - 3$

(3.5, -3)

202 CHAPTER 2: ANALYZING FUNCTIONS

For questions 17 – 18, identify the x-intercepts of each absolute value function.

17. $y = 3|2x - 8| - 6$
(3, 0) and (5, 0)

18. $y = \frac{1}{2}|-x + 1| + 4$
This graph does not have any x-intercepts.

For questions 19 – 21, use the functions shown below.

$f(x) = -(x - 2)$ $f(x) = -(x - 2)^2$ $h(x) = -|x - 2|$

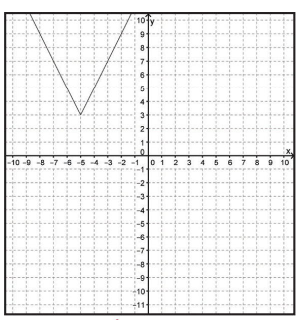
19. Find the x-intercept(s) of each function.
(2, 0)

20. Compare the x-intercept(s) of the functions and justify your answer.
See margin.

21. How would the x-intercept value(s) change if each function had a value of $d = 1$?
The 3 functions still have a common x-intercept of (3, 0), but g(x) and h(x) also have an x-intercept of (1, 0).

For questions 22 – 25, identify the domain and range of each function. Write the domain and range (A) as inequalities, (B) in set builder notation, and (C) in interval notation. Also, (D) identify the vertex of the function and (E) determine if the vertex is a maximum or minimum value.

22.



See margin.

23.

x	y
-1	-17
0	-11
1	-5
2	1
3	-5
4	-11
5	-17

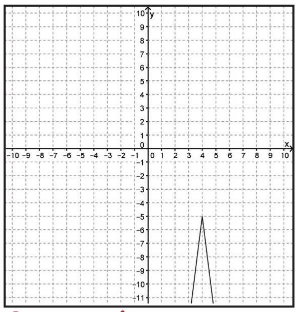
See margin.

24.

x	y
-3	2
-2.5	1
-2	0
-1.5	-1
-1	0
-0.5	1
0	2

See margin.

25.



See margin.

20. *The x-intercepts of all three functions are the same. They are all the same because c is the same value for all 3 functions and d = 0 for all 3 functions.*

22. *A. Domain: $-\infty < x < \infty$*
Range: $y \geq 3$

B. Domain: $\{x \mid x \in \mathbb{R}\}$
Range: $\{y \mid y \geq 3\}$

C. Domain: $(-\infty, \infty)$
Range: $[3, \infty)$

D. (-5, 3)

E. minimum value

23. *A. Domain: $-\infty < x < \infty$*
Range: $y \leq 1$

B. Domain: $\{x \mid x \in \mathbb{R}\}$
Range: $\{y \mid y \leq 1\}$

C. Domain: $(-\infty, \infty)$
Range: $(-\infty, 1]$

D. (2, 1)

E. maximum value

24. *A. Domain: $-\infty < x < \infty$*
Range: $y \geq -1$

B. Domain: $\{x \mid x \in \mathbb{R}\}$
Range: $\{y \mid y \geq -1\}$

C. Domain: $(-\infty, \infty)$
Range: $[-1, \infty)$

D. (–1.5, –1)

E. minimum value

25. *A. Domain: $-\infty < x < \infty$*
Range: $y \leq -5$

B. Domain: $\{x \mid x \in \mathbb{R}\}$
Range: $\{y \mid y \leq -5\}$

C. Domain: $(-\infty, \infty)$
Range: $(-\infty, -5]$

D. (4, –5)

E. maximum value