Modeling with Cubic Functions

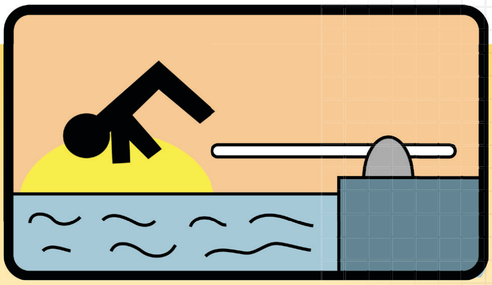
1.9

FOCUSING QUESTION How can you use finite differences to construct a cubic model for a data set?

LEARNING OUTCOMES

- I can use finite differences or common ratios to classify a function as linear, quadratic, cubic, or exponential when I am given a table of values.
- I can determine the cubic function from a table using finite differences, including any restrictions on the domain and range.
- I can use finite differences to write a cubic function that describes a data set.
- I can apply mathematics to problems that I see in everyday life, in society, and in the workplace.

ENGAGE



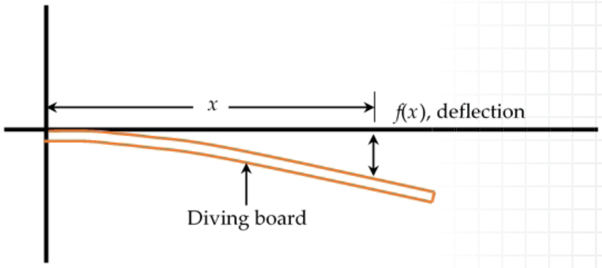
A diving board is a platform that someone can use to dive into a deep water pool. Diving boards typically have enough flexibility to allow a diver to bounce before leaving the board and entering the pool. Diving boards have a heel end on the pool deck, which is where the person steps onto the diving board, and a toe end that hangs over the pool. Diving boards also have a fulcrum that balances the board.

When a person stands on a diving board, it bends or deflects slightly due to the person's weight. What questions could you ask about the deflection of a diving board that could be answered by collecting data?

Work with a partner to give information you know about diving boards and this situation.

EXPLORE

When a person stands on a diving board, the diving board bends beneath the weight of the person. The farther the person stands from the fulcrum of the diving board, the more the diving board bends, or deflects, from the horizontal.



SUPPORTING ENGLISH LANGUAGE LEARNERS

For the Engage, pair students up so they may alternate giving information about what questions they could ask for the diving board situation. Encourage them to use key words and expressions related to functions and diving boards/swimming pools. Encourage students to think about both academic and social contexts (ELPS 3F).

TEKS

AR.2B Classify a function as linear, quadratic, cubic, and exponential when a function is represented tabularly using finite differences or common ratios as appropriate.

AR.2C Determine the function that models a given table of related values using finite differences and its restricted domain and range.

AR.2D Determine a function that models real-world data and mathematical contexts using finite differences such as the age of a tree and its circumference, figurative numbers, average velocity, and average acceleration.

MATHEMATICAL PROCESS SPOTLIGHT

AR.1A Apply mathematics to problems arising in everyday life, society, and the workplace.

ELPS

3F Ask and give information ranging from using a very limited bank of high-frequency, high-need, concrete vocabulary, including key words and expressions needed for basic communication in academic and social contexts, to using abstract and content-based vocabulary during extended speaking assignments.

VOCABULARY

cubic function, finite differences, maximum, minimum

MATERIALS

- graphing calculator

1. *No, because as x increases by 1 foot, the amount of deflection increases by a greater amount each time. The increase from a distance of 0 feet to 1 foot is 0.116 inches of deflection, but the increase from a distance of 1 foot to 2 feet is 0.332 inches of deflection.*

2. *No. The deflection between a distance of 1 foot and 2 feet increases by a factor of about 4, but the deflection between a distance of 2 feet and 3 feet increases by a factor of about 2.*

3. *See margin below.*

4. *See margin bottom of pages 118-119.*

5. *See margin and bottom of page 119.*

6. *Data set - domain: whole numbers, $0 \le x \le 5$; range: {0, 116, 448, 972, 1664, 2500}*

Function - domain: all real numbers; range: all real numbers

The domain and range of the data set are subsets of the domain and range of the function rule.

The domain and range of the data set are limited to whole numbers, but the domain and range of the function rule include all real numbers.

7-9. *See margin page 119.*

The table below shows the amount of deflection, in thousandths of an inch, when the same person stands x feet from the fulcrum of the diving board.

DISTANCE FROM FULCRUM (FT), x	DEFLECTION (0.001 IN.), $f(x)$
0	0
1	116
2	448
3	972
4	1664
5	2500

1. Does the data set appear to have a constant rate of change in deflection? Explain how you know.
See margin.

2. Does the amount of deflection appear to increase by the same factor each time the distance increases by 1 foot? Explain how you know.
See margin.

3. Calculate the finite differences between the deflection and the distance from the fulcrum. Do the data appear to be linear, quadratic, or cubic? How do you know?
The data appear to be cubic because the third differences are constant. See margin for details.

4. Use the patterns in the finite differences to write a function rule that describes the data set.
$f(x) = -4x^3 + 120x^2$
See margin for details.

5. Use a graphing calculator to graph the function rule over a scatterplot of the data. What do you notice about the function rule and the scatterplot?
The graph of the function rule connects each of the data points. See margin for graph.

6. Compare the domain and range of the data set and the domain and range of the function rule. How are they alike? How are they different?
See margin.

7. What will be the deflection if the person stands 10 feet from the fulcrum?
See margin.

8. Are the x-intercepts of the function included in your data set? Why or why not?
See margin.

9. What dimension of the diving board represents the greatest possible x-value that could be contained in the data set? What limit does that place on the domain of the data set?
See margin.

3.

DISTANCE FROM FULCRUM (FT), x	DEFLECTION (0.001 IN.), $f(x)$
0	0
1	116
2	448
3	972
4	1664
5	2500

+1, +1, +1, +1, +1

+116, +332, +524, +692, +836

+216, +192, +168, +144

-24, -24, -24

4. *For a cubic function of the form $f(x) = ax^3 + bx^2 + cx + d$,*

- *The value of d is the y-coordinate of the y-intercept, (0, d).*
- *The third difference is equal to 6a.*
- *The second difference between the first two pairs of y-values, x = 0 and x = 1, and x = 1 and x = 2, is equal to 6a + 2b.*
- *The first difference between the y-values for x = 0 and x = 1 is equal to a + b + c.*

REFLECT

- **How can you determine a cubic function model for a data set?**
  See margin.
- **What other factors could influence the deflection of the diving board? Explain how they would do so.**
  See margin.

EXPLAIN

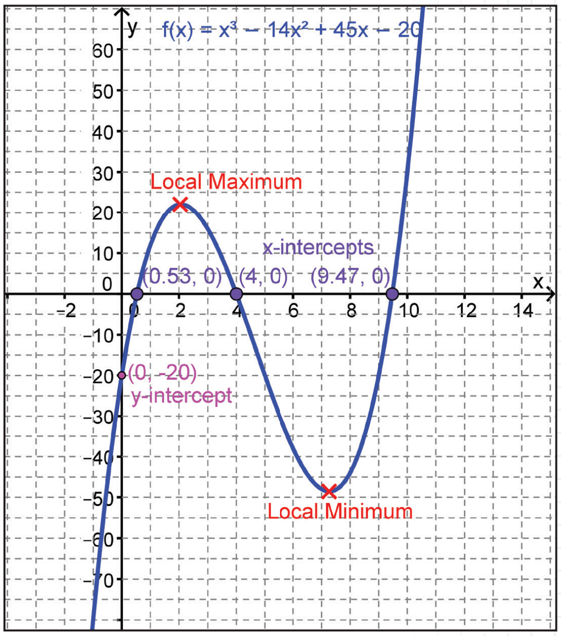
Cubic function models can be used to represent sets of mathematical and real-world data. Cubic functions have several attributes that should be considered when they are used for mathematical models.

Watch Explain and You Try It Videos

or click here

The domain and range of most cubic functions are all real numbers. Unlike quadratic functions, when you raise a negative number to the third power, you can have a negative number as a result.

Cubic functions have as many as 3 x-intercepts and 1 y-intercept. As with other function types, the x-intercepts represent x-values that generate a function value equal to 0. The y-intercept is also a starting point, or y-value when $x = 0$.



Quadratic functions have a maximum or minimum point at the vertex. These are absolute maximum or minimum values. A cubic function, however, may have what are called **local maximum** or **local minimum** values. These points are not the absolute maximum or minimum values for the function. Instead, they are maximum or minimum values for a nearby, or local, part of the function.

Let's look more closely at the data set and function model for the diving board problem.

5. *The graph of the function rule connects each of the data points.*

See graph below.

7. $f(x) = -4x^3 + 120x^2$

$f(10) = -4(10)^3 + 120(10)^2$

$f(10) = -4{,}000 + 12{,}000$

$f(10) = 8{,}000$

At a distance of 10 feet from the fulcrum, the diving board will deflect 8.000 inches.

8. *One x-intercept, (0, 0), is included in the data set because it makes senses that when the person stands 0 feet from the fulcrum that the diving board deflects 0.000 inches. The other x-intercept, (30, 0), is not included in the data set because it does not make sense that if a diving board were over 30 feet long that a person standing 30 feet from the fulcrum would not deflect the diving board.*

9. *The distance from the fulcrum to the end of the diving board is the greatest possible x-value that could be contained in the data set. This x-value represents the maximum for the domain of the data set.*

REFLECT ANSWERS:

See margin page 120.

4.

DISTANCE FROM FULCRUM (FT), x	DEFLECTION (0.001 IN.), $f(x)$
0	0
1	116
2	448
3	972
4	1664
5	2500

+1
+1
+1
+1
+1

d = 0

a + b + c = 116
(-4) + (120) + c = 116
116 + c = 116
c = 0

+116
+332
+524
+692
+836

+216
+192
+168
+144

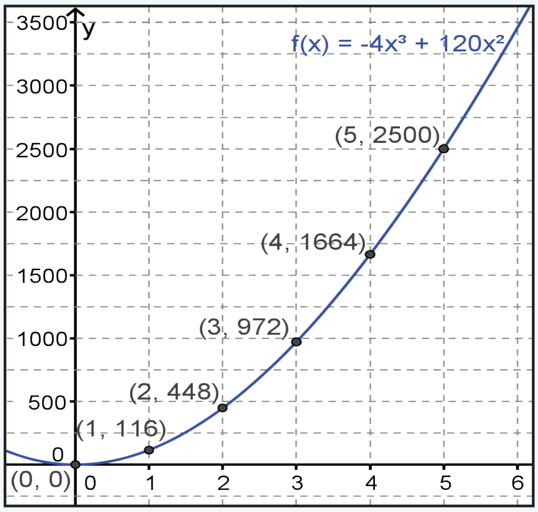
-24
-24
-24

6a + 2b = 216
6(-4) + 2b = 216
-24 + 2b = 216
2b = 240
b = 120

6a = -24
a = -4

$f(x) = -4x^3 + 120x^2$

5.



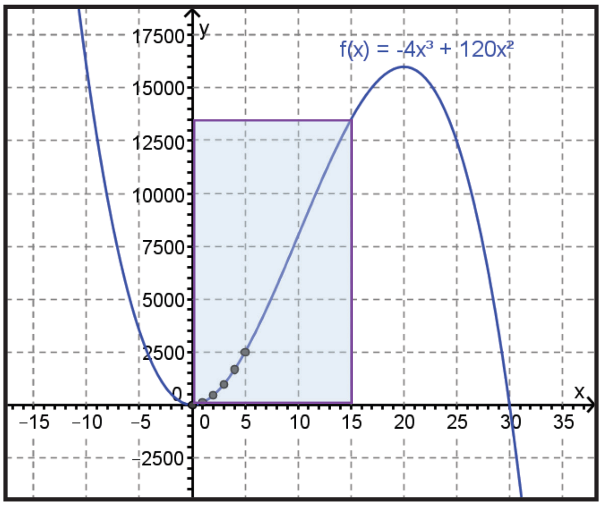
REFLECT ANSWERS:

Confirm that the third finite differences are constant. If they are, then work backward in the table if necessary to identify function values for $x = 0$ and $x = 1$. Use the patterns in the table to identify the values of a, b, c, and d to write a cubic function in standard (polynomial) form, $f(x) = ax^3 + bx^2 + cx + d$.

Answers may vary. Possible responses may include:

The weight of the person standing on the diving board influences the deflection. If the person weighs less than the person for whom the data was collected, then the deflection will be less. If the person weights more, then the deflection will be greater.

The flexibility or rigidity of the diving board influences the deflection. If the diving board is more rigid and doesn't bend as easily, then the deflection will be less.



In the diving board problem, the data points represent a small portion of the function. The shaded region on the graph shows the part of the function that would model the deflection of a diving board that has a distance of 15 feet between the end of the diving board and the fulcrum.

The function has two x-intercepts, one of which makes sense in the context of the diving board problem. When the person stands 0 feet from the fulcrum, you would expect the diving board to deflect 0 inches. However, it is not likely that a person standing 30 feet from the fulcrum would generate a deflection of 0 inches.

With **polynomial function models**, frequently the function model only represents the data set for a certain interval of the function. In the diving board problem, there is a local maximum at (20, 16,000). This means that a person standing 20 feet from the fulcrum causes a deflection of 16.000 inches. The function model for x-values greater than 20 then begins to decrease. It is not likely that a person standing farther than 20 feet from the fulcrum would generate less deflection, so the model likely does not represent the situation beyond a domain of 20 feet.

MODELING WITH CUBIC FUNCTIONS

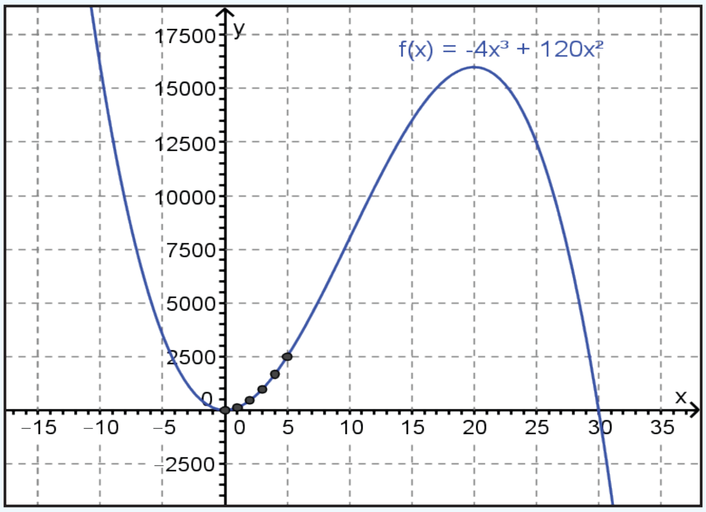
If the data set has a constant or approximately constant third finite difference, then a cubic function model may be appropriate for the data set.

The domain and range for the situation may be a subset of the domain and range of the cubic function model. Cubic functions can have intervals within their domain when they are increasing and intervals within their domain when they are decreasing.

TECHNOLOGY TIP

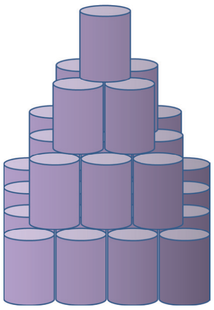
Some students may not be familiar with the shape of a graph of a cubic function. Use graphing technology such as a graphing calculator or a graphing app to allow students to explore the parent function of a cubic function, $f(x) = x^3$, as well as several other cubic functions that generate different shapes, including local maxima and minima.

For the function in the diving board situation, use graphing technology to zoom out and show students the shape of the complete graph.



EXAMPLE 1

A display of large juice cans takes the shape of a square-based pyramid with 1 can in the top level, 4 cans in the second level, 9 in the third level, 16 in the fourth level, etc. To predict the number of cans y for a display of x number of levels, determine if this situation represents a linear, quadratic, or cubic function.



LEVELS, x	PROCESS	NUMBER OF CANS, y
1	1	1
2	1 + 4	5
3	1 + 4 + 9	14
4	1 + 4 + 9 + 16	30
5	1 + 4 + 9 + 16 + 25	55
6	1 + 4 + 9 + 16 + 25 + 36	91

Use the data set to determine if the relationship is linear, quadratic, or cubic.

STEP 1 Consider the value of y, the number of cans, in the "zero" level of the display. The "zero" level would logically have no cans.

STEP 2 Determine the finite differences in values of x and the first, second, and third finite differences in values of $f(x)$.

LEVELS, x	PROCESS	NUMBER OF CANS, y
0	0	0
1	1	1
2	1 + 4	5
3	1 + 4 + 9	14
4	1 + 4 + 9 + 16	30
5	1 + 4 + 9 + 16 + 25	55
6	1 + 4 + 9 + 16 + 25 + 36	91

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta x = 5 - 4 = 1$

$\Delta x = 6 - 5 = 1$

+1, +4, +9, +16, +25, +36

+3, +5, +7, +9, +11

+2, +2, +2, +2

The set of data represents a cubic function because the differences in x are constant and the third finite differences between successive values of f(x) are constant.

ADDITIONAL EXAMPLES

Determine if the situations below represent a linear, quadratic, or cubic function.

1. A rocket is fired into the air. Its height above the ground, h, in meters at a given time, t, in seconds is shown in the table.

TIME (SECONDS), t	0	1	2	3	4
HEIGHT (METERS), h	6.5	79.6	142.9	196.4	240.1

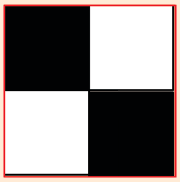
Quadratic

2. As a balloon was inflated, its radius, r, and volume, V, were recorded in the table shown.

RADIUS (IN), r	0	1	2	3	4	5
VOLUME (IN^3), V	0	4.19	33.51	113.1	268.08	523.6

Cubic

YOU TRY IT! #1



The number of squares on a checkerboard, including the number of 1 × 1 squares, 2 × 2 squares, 3 × 3 squares, and so on, are shown in the table. If the checkerboard is 1 × 1, there is only one square possible. In a 2 × 2 board, there are four 1 × 1 squares and one 4 × 4 square, for a total of five squares. The chart below shows the total number of squares contained in *two* checkerboards.

SIDE LENGTH OF BOARD IN SQUARES, x	TOTAL NUMBER OF SQUARES, $f(x)$
1	2
2	10
3	28
4	60
5	110
6	182
7	280
8	408

Determine whether the relationship is linear, quadratic, or cubic.

The set of data represents a cubic function because the differences in x values and the third finite differences in f(x) values are both constant.

EXAMPLE 2

Determine the cubic function rule to model the display of cans in Example 1. Compare the domain and range of the data set and the function.

Step 1 **Calculate the values of a, b, c, and d in the cubic function rule $f(x) = ax^3 + bx^2 + cx + d$ using the data in the table and finite differences.**

- There would be no cans in the "zero" level. The value for y when $x = 0$ is 0. So $d = 0$.
- The third finite difference in the set of data is 2. This equals $6a$. If $6a = 2$, then $a = \frac{1}{3}$.

ADDITIONAL EXAMPLE

Write the cubic function for Additional Example #2 on page 121.

$V = \frac{4}{3}\pi r^3$ *or* $V = 4.19r^3$

- The second difference between the first two pairs of y-values ($x = 0$ and $x = 1$; $x = 1$ and $x = 2$) is equal to $6a + 2b$.

$$\begin{aligned} 6a + 2b &= 3 \\ 2 + 2b &= 3 \end{aligned}$$

$$2b = 1 \text{ and } b = \frac{1}{2}$$

- The first difference between the y-values for $x = 0$ and $x = 1$ is equal to $a + b + c$.

$$\begin{aligned} a + b + c &= 1 \\ \frac{1}{3} + \frac{1}{2} + c &= 1 \\ c &= \frac{1}{6} \end{aligned}$$

STEP 3 **Write the cubic function rule with the values of a, b, c, and d:**

$$f(x) = \frac{1}{2}x^3 + \frac{1}{3}x^2 + \frac{1}{6}x$$

STEP 4 **Enter the x- and y-values from the table and then graph the function with technology. The graph connects the data points. The data points are limited to values for levels 1 to 8.**

- Considering the weight of the cans in several levels, it doesn't make sense to add more levels. Also, the function rule produces fractional parts of cans for many values of x.
- Data set - domain: whole numbers, $1 \le x \le 8$; range: {1, 5, 14, 30, 55, 91, 140, 204}
- Function - domain: all real numbers; range: all real numbers

The domain and range of the data set are subsets of the domain and range of the function rule.

The domain and range of the data set are limited to whole numbers, but the domain and range of the function rule include all real numbers.

ADDITIONAL EXAMPLES

Determine the cubic function rule to model the given data.

1.

x	0	1	2	3	4	5
y	13.75	14.75	29.75	76.75	173.75	338.75

$f(x) = 3x^3 – 2x^2 + \frac{1}{2}x + 13.25$

2.

x	0	1	2	3	4	5
y	-12	-19.5	-8	31.5	108	230.5

$f(x) = \frac{3}{2}x^3 + 5x^2 - 14x - 12$

YOU TRY IT! #2 ANSWER:

$f(x) = 4x^3 – 48x^2 + 144x$. A 2-inch cutout yields a tray with the greatest volume. In this situation, the domain includes whole numbers between 0 and 6, $\{x \mid x \in W, 0 < x < 6\}$ (There cannot be a cutout as great as half the width of the cardboard), and the range includes whole numbers, $0 < y < 128$.

INSTRUCTIONAL HINTS

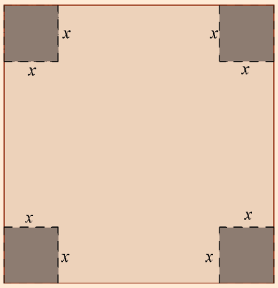
With every set of data, have students enter the function rule in their graphing calculators in order to see the graph. Once graphed, have students zoom out until they see the "bigger picture." Students might look at a small portion of a cubic function and mistake it for a quadratic function.

Help students process the differences and similarities in quadratic and cubic functions by writing about them.

Have students draw a Venn Diagram to compare and contrast the two types of functions.

YOU TRY IT! #2

A tray is made of a 12 in. **x** 12 in. square piece of cardboard with squares cut out of the four corners. The resulting sides are folded up and taped to form a square prism (open box). The volume of the tray related to the size of the squares cut from the corners.



SIZE OF SQUARE CUTS IN INCHES, x	PROCESS $(12 – 2x)(12 – 2x)(x)$	VOLUME IN CUBIC INCHES, $f(x)$
0	(12 – 0)(12 – 0)(0)	0
1	(12 – 2)(12 – 2)(1)	100
2	(12 – 4)(12 – 4)(2)	128
3	(12 – 6)(12 – 6)(3)	108
4	(12 – 8)(12 – 8)(4)	64
5	(12 – 10)(12 – 10)(5)	20
6	(12 – 12)(12 – 12)(6)	0

Generate a cubic function model for the volume of the tray. What size cutout produces the tray with the greatest volume? Identify the domain and range that most appropriately models the data.

See margin.

EXAMPLE 3

A set of graduated cubes comes with a large industrial scale. The cubes are numbered according to increasing size. The weights of some of the cubes are given in the table. Determine if the data set represents a linear, quadratic, or cubic function. Identify the domain and range of the model that most appropriately models the data.

CUBE NUMBER, x	WEIGHT IN OUNCES, $f(x)$
2	12.5
4	34.3
6	72.9
8	133.1
10	219.7
12	337.5

STEP 1 Determine the finite differences in values of x and the first, second, and third finite differences in values of $f(x)$.

	CUBE NUMBER, x	WEIGHT IN OUNCES, $f(x)$			
	2	12.5			
$\Delta x = 4 - 2 = 2$			+21.8		
	4	34.3		+16.8	
$\Delta x = 6 - 4 = 2$			+38.6		+4.8
	6	72.9		+21.6	
$\Delta x = 8 - 6 = 2$			+60.2		+4.8
	8	133.1		+26.4	
$\Delta x = 10 - 8 = 2$			+86.6		+4.8
	10	219.7		+31.2	
$\Delta x = 12 - 10 = 2$			+117.8		
	12	337.5			

The set of data represents a cubic function because the differences in x are constant, although the Δx is 2, not 1, and the third finite differences between successive values of $f(x)$ are constant.

STEP 2 From the information, it is unclear if there are larger or odd-numbered cubes.

The set of data represents a cubic function.
The domain is a subset of whole numbers, $0 < x \leq 12$.
The range is a subset of real numbers, $0 < y \leq 337.5$.

ADDITIONAL EXAMPLES

Determine the cubic function rule to model the data sets below.

1.

x	0	1	2	3	4	5
y	0	-0.7	-52.4	-227.1	-596.8	-1233.5

$f(x) = -12x^3 + 10.5x^2 + 4.5x$

2.

x	0	1	2	3	4	5
y	0	42.33	86.67	135	189.33	251.67

$f(x) = \frac{1}{3}x^3 + 42x$

YOU TRY IT! #3 ANSWER:

The differences in the x-values are constant. The third finite differences, which average 0.33, are approximately constant, so the data set represents a cubic function model. The domain represents the annual average wind speeds in the USA, so the domain of the data set is the subset of real numbers, $4 \leq x \leq 10$. *The annual energy output reflects the data collected for those average wind speeds, so the range of the set is the subset of real numbers,* $3.40 \leq f(x) \leq 53.12$.

YOU TRY IT! #3

The power, y, (in kilowatts) generated by a wind turbine is related to the wind speed. Determine if the data set represents a linear, quadratic, or cubic function. Identify the domain and range of the model that most appropriately models the data.

AVERAGE ANNUAL WIND SPEED IN THE USA IN METERS PER SECOND, x	ANNUAL ENERGY OUTPUT IN KWH/YEAR, $f(x)$
4	3.40
5	6.64
6	11.47
7	18.22
8	27.25
9	38.75
10	53.12

Data Source: National Renewal Energy Laboratory and Energy.gov

See margin.

PRACTICE/HOMEWORK

For questions 1 – 6, use finite differences to determine if the data sets shown in the tables below represent a linear, exponential, quadratic, or cubic function.

1.

x	y
0	3
1	6
2	12
3	24
4	48
5	96

Exponential

2.

x	y
0	-6
1	1
2	16
3	39
4	70
5	109

Quadratic

3.

x	y
0	2.25
1	8.75
2	15.25
3	21.75
4	28.25
5	34.75

Linear

4.

x	y
0	5
1	10
2	35
3	92
4	193
5	350

Cubic

5.

x	y
0	20
1	50
2	125
3	312.5
4	781.25
5	1953.125

Exponential

6.

x	y
0	-8
1	3
2	44
3	145
4	336
5	647

Cubic

For questions 7 – 12, the data sets shown in the tables represent cubic functions. Use finite differences to determine the function that relates the variables.

7.

x	y
0	-1
1	0
2	11
3	50
4	135
5	284

$f(x) = 3x^3 - 4x^2 + 2x - 1$

8.

x	y
0	3
1	7
2	1
3	-27
4	-89
5	-197

$f(x) = -2x^3 + x^2 + 5x + 3$

9.

x	y
0	0
1	14
2	72
3	198
4	416
5	750

$f(x) = 4x^3 + 10x^2$

10.

x	y
0	-4
1	-9
2	-48
3	-157
4	-372
5	-729

$f(x) = -6x^3 + x^2 - 4$

11.

x	y
0	1
1	6
2	15
3	31
4	57
5	96

$f(x) = \frac{1}{2}x^3 + \frac{1}{2}x^2 + 4x + 1$

12.

x	y
0	-9
1	-3
2	39
3	153
4	375
5	741

$f(x) = 6x^3 - 9$

For questions 13 – 17 use the scenario below.

GEOMETRY

A box is created from a 20-inch by 24-inch rectangular piece of cardboard by cutting congruent squares from each corner. The squares are cut in 1-inch increments. The resulting sides are folded up and taped to form a rectangular prism (open box). The volume of the box is a function of the side length of the square removed from each corner. The table below relates the volume of the box to the side length of the square.

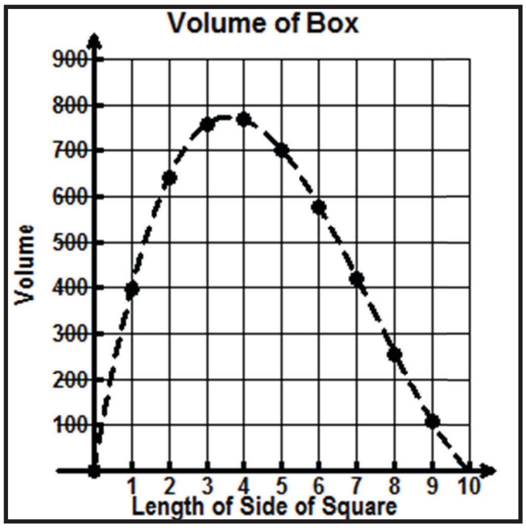
SIDE LENGTH x	VOLUME y
0	0
1	396
2	640
3	756
4	768
5	700
6	576
7	420
8	256
9	108

13. Generate a cubic function model for the volume of tray when given the side length of the square cut from each of the corners.
$y = 4x^3 – 88x^2 + 480x$

14. What side length of the square produces a tray with the greatest volume?
4 inches

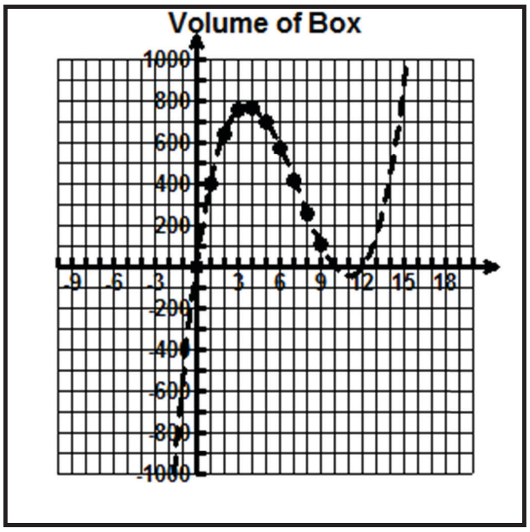
15. The graph to the right represents the data in the table and the function that models the table.

What is the domain and range of this situation?
Domain = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}
Range = {0, 108, 256, 396, 420, 576, 640, 700, 756, 768}



16. The graph to the right represents the data in the table and the function that models the table, but has been graphed with a different window setting.

Why are there no individual points plotted on the graph for $x < 0$ and $x > 10$?
The side length of the cut-out square cannot be a negative number and it also cannot be greater than half the length or width of the rectangle.



17. Why does the domain and range only contain whole numbers?
The squares cut from each corner of the rectangular piece of cardboard are cut in 1-inch increments.

For questions 18 – 22 use the scenario below.

CRITICAL THINKING

An employee at a toy store is creating a display of soccer balls in the shape of a tetrahedron, or an equilateral triangular pyramid.

The table below shows the total number of soccer balls at each level of the display, with Level 1 being at the top of the display.

LEVEL, x	TOTAL NUMBER OF SOCCER BALLS, y
1	1
2	4
3	10
4	20
5	35
6	56

18. Write a function using finite differences that models the data in the table.
$f(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{3}x$

19. What does the domain of the function represent in the situation?
The level number

20. What does the range of the function represent in the situation?
The total number of soccer balls used to build the display through that level.

21. Is 2.5 an element in the domain of this situation? Why or why not?
No; there cannot be 0.5 of a level.

22. How many soccer balls would be needed to build a display 10 levels high?
220 soccer balls