# 1.8 Writing Cubic Functions

**FOCUSING QUESTION** What are the characteristics of a cubic function?

## LEARNING OUTCOMES

- I can determine patterns that identify a cubic function from its related finite differences.
- I can determine the cubic function from a table using finite differences, including any restrictions on the domain and range.
- I can use finite differences to determine a cubic function that models a mathematical context.
- I can analyze patterns to connect the table to a function rule and communicate the cubic pattern as a function rule.

## ENGAGE

A tomato sauce can is in the shape of a cylinder and the diameter of the base is equal to the height of the can. Generate a sequence showing the volumes of a series of cans with a radius of 1 inch, 2 inches, 3 inches, and 4 inches. What patterns do you notice in the sequence?

***See margin.***

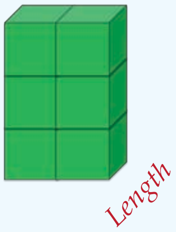
## EXPLORE

The volume of a prism is found using the area of the base and the height of the prism, $V = Bh$. If the prism is a rectangular prism, then the base is a rectangle and its area is the product of the length and width of the rectangle, $A = lw$. Combining these formulas generates a formula you can use to determine the volume of a rectangular prims, $V = lwh$.

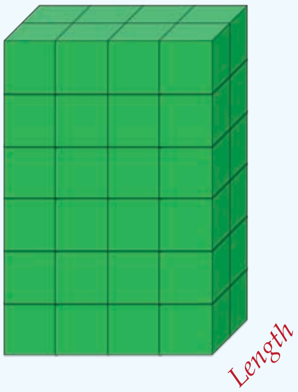
Use cubes to build the first two terms in a sequence of rectangular prisms. For this sequence, the term number is the length of the prism, the width of the prism is double the term number, and the height of the prism is triple the term number.

**1.** Sketch the first two terms that you built with the cubes.
***See margin.***

**1.** *Term 1*



*Term 2*



## TEKS

**AR.2A** Determine the patterns that identify the relationship between a function and its common ratio or related finite differences as appropriate, including linear, quadratic, cubic, and exponential functions.

**AR.2C** Determine the function that models a given table of related values using finite differences and its restricted domain and range.

**AR.2D** Determine a function that models real-world data and mathematical contexts using finite differences such as the age of a tree and its circumference, figurative numbers, average velocity, and average acceleration.

## MATHEMATICAL PROCESS SPOTLIGHT

**AR.1F** Analyze mathematical relationships to connect and communicate mathematical ideas.

## ELPS

**3D** Speak using grade-level content area vocabulary in context to internalize new English words and build academic language proficiency.

## VOCABULARY

finite differences, cubic function

## MATERIALS

- 60 building blocks per student group

### ENGAGE ANSWER:

*Possible answer:* $2\pi$, $16\pi$, $54\pi$, $128\pi$

*The number that is multiplied by* $\pi$ *is twice the value of the radius cubed.*

## SUPPORTING ENGLISH LANGUAGE LEARNERS

Encourage all students, especially English language learners, to speak using appropriate mathematics vocabulary (ELPS: 3D). Doing so helps them internalize new English words and make connections between old and new content.

**2.** Complete a table like the one shown using the relationships among the dimensions of the prism (length = $x$, width = $2x$, and height = $3x$).

| TERM NUMBER | PROCESS | VOLUME |
|---|---|---|
| 1 | 1(2)(3) | 6 |
| 2 | ***2(4)(6)*** | ***48*** |
| 3 | ***3(6)(9)*** | ***162*** |
| 4 | ***4(8)(12)*** | ***384*** |
| 5 | ***5(10)(15)*** | ***750*** |
| 6 | ***6(12)(18)*** | ***1296*** |

**3.** Does the data set follow a linear or an exponential function? Explain your reasoning.
***See margin.***

**4.** Calculate the second finite differences. What do you notice?
***The second differences are not constant. See margin for details.***

**5.** Calculate the third finite differences. What do you notice?
***The third differences are constant. See margin for details.***

**6.** Use the relationships among the dimensions that you were originally given to calculate the volume of a rectangular prism with a length of $x$ units. What type of function does this appear to be?
***$V = x(2x)(3x) = 6x^3$***
***The volume equation is a cubic function.***

## REFLECT

- **A cubic function is a function of the form $f(x) = ax^3 + bx^2 + cx + d$. What is the degree of this function (i.e., the power of the greatest exponent)?**
  ***A cubic function has a degree of 3.***

- **A linear function contains a polynomial with degree one ($mx + b$) and a quadratic function contains a polynomial with degree two ($ax^2 + bx + c$). What relationship is there between the degree of the polynomial and the level of finite differences that are constant?**
  ***See margin.***

**REFLECT ANSWERS:**

*The level of finite differences that are constant is the same as the degree of the polynomial (linear: degree one and the first differences are constant; quadratic: degree two and the second differences are constant; cubic: degree three and the third differences are constant)*

**3.**

| TERM NUMBER | PROCESS | VOLUME |
|---|---|---|
| 1 | 1(2)(3) | 6 |
| 2 | ***2(4)(6)*** | ***48*** |
| 3 | ***3(6)(9)*** | ***162*** |
| 4 | ***4(8)(12)*** | ***384*** |
| 5 | ***5(10)(15)*** | ***750*** |
| 6 | ***6(12)(18)*** | ***1296*** |

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta x = 5 - 4 = 1$

$\Delta x = 6 - 5 = 1$

***$\Delta y = 42$***

***$\Delta y = 114$***

***$\Delta y = 222$***

***$\Delta y = 366$***

***$\Delta y = 546$***

*In the table, $\Delta x = 1$. The data set is not linear because the first differences in the volume are not constant.*

| TERM NUMBER | PROCESS | VOLUME |
|---|---|---|
| 1 | 1(2)(3) | 6 |
| 2 | ***2(4)(6)*** | ***48*** |
| 3 | ***3(6)(9)*** | ***162*** |
| 4 | ***4(8)(12)*** | ***384*** |
| 5 | ***5(10)(15)*** | ***750*** |
| 6 | ***6(12)(18)*** | ***1296*** |

$\frac{y_n}{y_{n-1}} = \frac{48}{6} = 8$

$\frac{y_n}{y_{n-1}} = \frac{162}{48} = 3.375$

$\frac{y_n}{y_{n-1}} = \frac{384}{162} \approx 2.37$

$\frac{y_n}{y_{n-1}} = \frac{750}{384} \approx 1.95$

$\frac{y_n}{y_{n-1}} = \frac{1296}{750} = 1.728$

*The data set is not exponential because the successive ratios are not constant*

## EXPLAIN

In a linear function, the first finite differences, or the difference between consecutive values of the dependent variable, are constant. For a quadratic function, the second finite differences are constant. In a cubic function, the third finite differences are constant.

Let's look more closely at a cubic function. The table below shows the relationship between $x$ and $f(x)$. In a cubic function written in polynomial or standard, $f(x) = ax^3 + bx^2 + cx + d$.

There are many forms of a cubic function. Polynomial form, also called standard form, expresses a function as a polynomial with exponents in decreasing order.

$$f(x) = ax^3 + bx^2 + cx + d$$

In standard form, $a$, $b$, $c$, and $d$ are rational numbers.

| $x$ | PROCESS | $y = f(x)$ |
|---|---|---|
| 0 | $a(0)^3 + b(0)^2 + c(0) + d$ | $d$ |
| 1 | $a(1)^3 + b(1)^2 + c(1) + d$ | $a + b + c + d$ |
| 2 | $a(2)^3 + b(2)^2 + c(2) + d$ | $8a + 4b + 2c + d$ |
| 3 | $a(3)^3 + b(3)^2 + c(3) + d$ | $27a + 9b + 3c + d$ |
| 4 | $a(4)^3 + b(4)^2 + c(4) + d$ | $64a + 16b + 4c + d$ |
| 5 | $a(5)^3 + b(5)^2 + c(5) + d$ | $125a + 25b + 5c + d$ |

$\Delta x = 1 - 0 = 1$
$\Delta x = 2 - 1 = 1$
$\Delta x = 3 - 2 = 1$
$\Delta x = 4 - 3 = 1$
$\Delta x = 5 - 4 = 1$

$\Delta y = a + b + c$
$\Delta y = 7a + 3b + c$
$\Delta y = 19a + 5b + c$
$\Delta y = 37a + 7b + c$
$\Delta y = 61a + 9b + c$

The first differences are not constant, so let's look at the second differences.

| $x$ | PROCESS | $y = f(x)$ |
|---|---|---|
| 0 | $a(0)^3 + b(0)^2 + c(0) + d$ | $d$ |
| 1 | $a(1)^3 + b(1)^2 + c(1) + d$ | $a + b + c + d$ |
| 2 | $a(2)^3 + b(2)^2 + c(2) + d$ | $8a + 4b + 2c + d$ |
| 3 | $a(3)^3 + b(3)^2 + c(3) + d$ | $27a + 9b + 3c + d$ |
| 4 | $a(4)^3 + b(4)^2 + c(4) + d$ | $64a + 16b + 4c + d$ |
| 5 | $a(5)^3 + b(5)^2 + c(5) + d$ | $125a + 25b + 5c + d$ |

$\Delta x = 1 - 0 = 1$
$\Delta x = 2 - 1 = 1$
$\Delta x = 3 - 2 = 1$
$\Delta x = 4 - 3 = 1$
$\Delta x = 5 - 4 = 1$

$\Delta y = a + b + c$
$\Delta y = 7a + 3b + c$
$\Delta y = 19a + 5b + c$
$\Delta y = 37a + 7b + c$
$\Delta y = 61a + 9b + c$

$6a + 2b$
$12a + 2b$
$18a + 2b$
$24a + 2b$

**4.**

| TERM NUMBER | PROCESS | VOLUME |
|---|---|---|
| 1 | 1(2)(3) | 6 |
| 2 | 2(4)(6) | 48 |
| 3 | 3(6)(9) | 162 |
| 4 | 4(8)(12) | 384 |
| 5 | 5(10)(15) | 750 |
| 6 | 6(12)(18) | 1296 |

$\Delta x = 2 - 1 = 1$
$\Delta x = 3 - 2 = 1$
$\Delta x = 4 - 3 = 1$
$\Delta x = 5 - 4 = 1$
$\Delta x = 6 - 5 = 1$

$\Delta y = 42$
$\Delta y = 114$
$\Delta y = 222$
$\Delta y = 366$
$\Delta y = 546$

72
108
144
180

*The second differences are not constant.*

**5.**

| TERM NUMBER | PROCESS | VOLUME |
|---|---|---|
| 1 | 1(2)(3) | 6 |
| 2 | 2(4)(6) | 48 |
| 3 | 3(6)(9) | 162 |
| 4 | 4(8)(12) | 384 |
| 5 | 5(10)(15) | 750 |
| 6 | 6(12)(18) | 1296 |

$\Delta x = 2 - 1 = 1$
$\Delta x = 3 - 2 = 1$
$\Delta x = 4 - 3 = 1$
$\Delta x = 5 - 4 = 1$
$\Delta x = 6 - 5 = 1$

$\Delta y = 42$
$\Delta y = 114$
$\Delta y = 222$
$\Delta y = 366$
$\Delta y = 546$

72
108
144
180

36
36
36

*The third differences are constant.*

## INTEGRATE TECHNOLOGY

Use technology such as a graphing calculator or spreadsheet app on a display screen to show students how, no matter the numbers present in the cubic function, the third differences will always be constant.

The second differences are not constant, either. But, you can see some patterns emerging. Notice that every time you take another round of finite differences, the last constant term drops off because it is subtracted out. For the second differences, the coefficients of the a term are multiples of 6. Also, each second difference has the same $b$ term, $2b$. Let's calculate the third differences.

*Watch Explain and You Try It Videos*

*or click here*

| $x$ | PROCESS | $y = f(x)$ |
|---|---|---|
| 0 | $a(0)^3 + b(0)^2 + c(0) + d$ | $d$ |
| 1 | $a(1)^3 + b(1)^2 + c(1) + d$ | $a + b + c + d$ |
| 2 | $a(2)^3 + b(2)^2 + c(2) + d$ | $8a + 4b + 2c + d$ |
| 3 | $a(3)^3 + b(3)^2 + c(3) + d$ | $27a + 9b + 3c + d$ |
| 4 | $a(4)^3 + b(4)^2 + c(4) + d$ | $64a + 16b + 4c + d$ |
| 5 | $a(5)^3 + b(5)^2 + c(5) + d$ | $125a + 25b + 5c + d$ |

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta x = 5 - 4 = 1$

$\Delta y = a + b + c$

$\Delta y = 7a + 3b + c$

$\Delta y = 19a + 5b + c$

$\Delta y = 37a + 7b + c$

$\Delta y = 61a + 9b + c$

$6a + 2b$

$12a + 2b$

$18a + 2b$

$24a + 2b$

$6a$

$6a$

$6a$

The third differences are, indeed, constant. Each third difference is $6a$. You can use patterns from the table to determine the cubic function from the table of data.

- The value of $d$ is the $y$-coordinate of the $y$-intercept, $(0, d)$.
- The third difference is equal to $6a$.
- The second difference between the first two pairs of $y$-values is equal to $6a + 2b$.
- The first difference between the $y$-values for $x = 0$ and $x = 1$ is equal to $a + b + c$.

### FINITE DIFFERENCES AND CUBIC FUNCTIONS

In a cubic function, the third differences between successive $y$-values are constant if the differences between successive $x$-values, $\Delta x$, are also constant.

If the third differences between consecutive $y$-values in a table of values are constant, then the values represent a cubic function.

The formulas for finding the values of $a$, $b$, $c$, and $d$ to write the cubic function only work when $\Delta x = 1$. When $\Delta x \neq 1$, there are other formulas that can be used to determine the values of $a$, $b$, $c$, and $d$ for the cubic function.

## EXAMPLE 1

Use the table from the Explore activity to determine the cubic function rule for the volume of a set of rectangular prisms with width = $x$, length = $2x$, and height = $3x$.

| TERM NUMBER, $x$ | PROCESS | VOLUME, $y$ |
|---|---|---|
| 1 | 1(2)(3) | 6 |
| 2 | 2(4)(6) | 48 |
| 3 | 3(6)(9) | 162 |
| 4 | 4(8)(12) | 384 |
| 5 | 5(10)(15) | 750 |
| 6 | 6(12)(18) | 1296 |

**STEP 1** Analyze the first, second, and third finite differences to calculate the values of $a$, $b$, $c$, and $d$ for the function rule $f(x) = ax^3 + bx^2 + cx + d$. Use the finite differences to calculate the value of $y$ for $x = 0$.

| TERM NUMBER, $x$ | PROCESS | VOLUME, $y$ |
|---|---|---|
| 0 | 0(0)(0) | 0 |
| 1 | 1(2)(3) | 6 |
| 2 | 2(4)(6) | 48 |
| 3 | 3(6)(9) | 162 |
| 4 | 4(8)(12) | 384 |
| 5 | 5(10)(15) | 750 |
| 6 | 6(12)(18) | 1296 |

$\Delta x = 1 - 0 = 1$
$\Delta x = 2 - 1 = 1$
$\Delta x = 3 - 2 = 1$
$\Delta x = 4 - 3 = 1$
$\Delta x = 5 - 4 = 1$
$\Delta x = 6 - 5 = 1$

$\Delta y = 6$
$\Delta y = 42$
$\Delta y = 114$
$\Delta y = 222$
$\Delta y = 366$
$\Delta y = 546$

36
72
108
144
180

36
36
36
36

**STEP 2** The value of $d$ is the $y$-coordinate of the $y$-intercept, (0, $d$), and for this data set, $d = 0$.

**STEP 3** The third difference is equal to $6a$. Since $6a = 36$, $a = 6$.

## QUESTIONING STRATEGIES

Have students read the scenario for Example 1 again after completing the steps to write the cubic function rule. Draw their attention to the process column.

- How did the scenario translate into the cubic function rule?
- How is the cubic function rule reflected in the process column on the table?
- Could you have written this cubic function rule from the scenario and table without the steps below?

## ADDITIONAL EXAMPLE

Veronica makes cylindrical candles to sell on a popular crafting website. To order the proper amount of wax, she calculates the volume of each candle. Use the table below to determine the cubic function rule for the volume of her candles with radius = $x$ and height = $3x$.

$y = 3(x^3)\ \pi$

| TERM NUMBER, $x$ | PROCESS | VOLUME (EXACT), $y$ | VOLUME (ESTIMATED), $y$ |
|---|---|---|---|
| 0 | $\Pi\ (0^2)(3 \cdot 0)$ | 0 | 0 |
| 1 | $\Pi\ (1^2)(3 \cdot 1)$ | 3 Π | 9.42 |
| 2 | $\Pi\ (2^2)(3 \cdot 2)$ | 24 Π | 75.4 |
| 3 | $\Pi\ (3^2)(3 \cdot 3)$ | 81 Π | 254.47 |
| 4 | $\Pi\ (4^2)(3 \cdot 4)$ | 192 Π | 603.19 |
| 5 | $\Pi\ (5^2)(3 \cdot 5)$ | 375 Π | 1178.1 |

**STEP 4** The second difference between the first two pairs of $y$-values ($x = 0$ and $x = 1$; $x = 1$ and $x = 2$) is $6a + 2b$.

$$\begin{aligned} 6a + 2b &= 36 \\ 36 + 2b &= 36 \\ 2b &= 0 \\ b &= 0 \end{aligned}$$

**STEP 5** The first difference between the $y$-values for $x = 0$ and $x = 1$ is equal to $a + b + c$.

$$\begin{aligned} a + b + c &= 6 \\ 6 + 0 + c &= 6 \\ c &= 0 \end{aligned}$$

**STEP 6** Write the cubic function rule.

*$y = 6x^3 + 0x^2 + 0x + 0$, or simply $y = 6x^3$.*

## YOU TRY IT! #1

Popcorn is sold in boxes in the shape of square prisms. The dimensions of the boxes and their volumes are shown in the table. Write an equation to describe the volume, $y$, related to the width of the box, $x$, and verify it with a function rule using the finite differences in the table.

| BOX NUMBER | BOX WIDTH IN INCHES, $x$ | WIDTH, LENGTH, AND HEIGHT IN INCHES | VOLUME IN CUBIC INCHES, $y$ |
|---|---|---|---|
| 1 (SAMPLER) | 1 | 1(1)(2) | 2 |
| 2 (KID'S) | 2 | 2(2)(4) | 16 |
| 3 (SMALL) | 3 | 3(3)(6) | 54 |
| 4 (MEDIUM) | 4 | 4(4)(8) | 128 |
| 5 (LARGE) | 5 | 5(5)(10) | 250 |
| 6 (SUPER) | 6 | 6(6)(12) | 432 |

***The formula $V = lwh$ becomes $y = x(x)(2x) = 2x^3$. The function rule, using finite differences in the table, is $f(x) = 2x^3 + 0x^2 + 0x + 0$, or simply $y = 2x^3$.***

### INSTRUCTIONAL HINT

When completing YOU TRY IT #1 some students may struggle with writing this function rule. If so, draw their attention back to #6 in the Explore exercise, and encourage them to focus on the "width, length, and height in inches" column of the table.

## EXAMPLE 2

Determine whether the data set shown is linear, quadratic, exponential, or cubic.

| $x$ | $y$ |
|---|---|
| -1 | 34 |
| 0 | 50 |
| 1 | 56 |
| 2 | 58 |
| 3 | 62 |
| 4 | 74 |
| 5 | 100 |

**STEP 1** Determine the first, second, and third differences between successive $x$-values and successive $y$-values.

| $x$ | $y$ |
|---|---|
| -1 | 34 |
| 0 | 50 |
| 1 | 56 |
| 2 | 58 |
| 3 | 62 |
| 4 | 74 |
| 5 | 100 |

$\Delta x = 0 - -1 = 1$

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta x = 5 - 4 = 1$

$\Delta y = 50 - 34 = 16$

$\Delta y = 56 - 50 = 6$

$\Delta y = 58 - 56 = 2$

$\Delta y = 62 - 58 = 4$

$\Delta y = 74 - 62 = 12$

$\Delta y = 100 - 74 = 26$

-10, -4, +2, +8, +14

+6, +6, +6, +6

## ADDITIONAL EXAMPLES

Determine whether the data sets shown are linear, quadratic, exponential, or cubic.

**1.**

| | | | | | |
|---|---|---|---|---|---|
| $x$ | 0 | 1 | 2 | 3 | 4 |
| $y$ | -8 | -2 | 18 | 70 | 172 |

*Cubic*

**2.**

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | -2 | -1 | 0 | 1 | 2 | 3 |
| $y$ | 0.125 | 0.5 | 2 | 8 | 32 | 128 |

*Exponential*

## ADDITIONAL EXAMPLES

Determine whether the data sets shown are linear, quadratic, exponential, or cubic.

**1.**

| $x$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| $y$ | 15 | 23.5 | 36 | 52.5 | 73 |

*Quadratic*

**2.**

| $x$ | -3 | 0 | 3 | 6 | 9 |
|---|---|---|---|---|---|
| $y$ | 8 | 6 | 4 | 2 | 0 |

*Linear*

## INSTRUCTIONAL HINT

When learning quadratic models, the Instructional Strategies suggested that students create a flowchart for testing data to determine the type of function represented by a set of data. Have students return to the flowchart they created to add testing for cubic functions.

**STEP 2** Determine the ratios between successive $y$-values.

| $x$ | $y$ |
|---|---|
| -1 | 34 |
| 0 | 50 |
| 1 | 56 |
| 2 | 58 |
| 3 | 62 |
| 4 | 74 |
| 5 | 100 |

$$\frac{y_n}{y_{n-1}} = \frac{50}{34} \approx 1.47$$

$$\frac{y_n}{y_{n-1}} = \frac{56}{50} \approx 1.12$$

$$\frac{y_n}{y_{n-1}} = \frac{58}{56} \approx 1.04$$

$$\frac{y_n}{y_{n-1}} = \frac{62}{58} \approx 1.07$$

$$\frac{y_n}{y_{n-1}} = \frac{74}{62} \approx 1.19$$

$$\frac{y_n}{y_{n-1}} = \frac{100}{74} \approx 1.35$$

**STEP 3** Determine whether the set of data represents a linear, quadratic, exponential, or cubic function.

*The differences in x, Δx, are all 1, so they are constant.*

*The first differences are not constant, so the set of data does not represent a linear function.*

*The second differences are not constant, so the set of data does not represent a quadratic function.*

*The ratios between successive y-values are not constant, so the set of data does not represent an exponential function.*

*The third differences are constant, so the set of data represents a cubic function.*

## YOU TRY IT! #2

Does the set of data shown below represent a cubic function? Justify your answer.

| $x$ | $y$ |
|---|---|
| 1 | 2 |
| 2 | 5 |
| 3 | 11 |
| 4 | 23 |
| 5 | 47 |
| 6 | 99 |
| 7 | 191 |

***See margin.***

## EXAMPLE 3

Write a cubic function for the values in the table.

| $x$ | $y$ |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | 2 |
| 3 | 4 |
| 4 | 8 |
| 5 | 15 |
| 6 | 26 |

### YOU TRY IT! #2 ANSWER:

*No, the set of data does not represent a cubic function, because the differences in x are constant but the third finite differences in y are not constant.*

### ADDITIONAL EXAMPLES

Write the function rules for the tables in the Additional Examples on pages 109 and 110.

**pg. 109, AE #1:**
$y = 3x^3 - 2x^2 + 5x - 8$

**pg. 109, AE #2:**
$y = 2(4)^x$

**pg. 110, AE #1:**
$y = 2x^2 - 1.5x + 10$

**pg. 110, AE #2:**
$y = -\frac{2}{3}x + 6$

## INSTRUCTIONAL HINT

Have students add the steps for writing a cubic function to their flowchart (page 110). Encourage students to use their flowchart to study for the test and guide their practice.

## ADDITIONAL EXAMPLES

Write a cubic function for the values in the tables.

**1.**

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $y$ | 12 | 11.33 | 20.67 | 42 | 77.33 | 128.67 |

$y = \frac{1}{3}x^3 + 4x^2 - 5x + 12$

**2.**

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $y$ | 0 | -6.5 | -45 | -139.5 | -314 | -592.5 |

$y = -4x^3 - 4x^2 + \frac{3}{2}x$

**STEP 1** Determine the first differences between successive $x$-values and the third finite differences in successive $y$-values.

| | $x$ | $y$ | | | |
|---|---|---|---|---|---|
| | 0 | 0 | | | |
| $\Delta x = 1 - 0 = 1$ | | | $\Delta y = 1 - 0 = 1$ | | |
| | 1 | 1 | | 0 | |
| $\Delta x = 2 - 1 = 1$ | | | $\Delta y = 2 - 1 = 1$ | | 1 |
| | 2 | 2 | | 1 | |
| $\Delta x = 3 - 2 = 1$ | | | $\Delta y = 4 - 2 = 2$ | | 1 |
| | 3 | 4 | | 2 | |
| $\Delta x = 4 - 3 = 1$ | | | $\Delta y = 8 - 4 = 4$ | | 1 |
| | 4 | 8 | | 3 | |
| $\Delta x = 5 - 4 = 1$ | | | $\Delta y = 15 - 8 = 7$ | | 1 |
| | 5 | 15 | | 4 | |
| $\Delta x = 6 - 5 = 1$ | | | $\Delta y = 26 - 15 = 11$ | | |
| | 6 | 26 | | | |

**STEP 2** Calculate the values for $a$, $b$, $c$, and $d$ in $f(x) = ax^3 + bx^2 + cx + d$.

- The value of $d$ is the $y$-coordinate of the $y$-intercept, $(0, d)$. For this data set, $d = 0$.
- The third difference is equal to $6a$. Since $6a = 1$, $a = \frac{1}{6}$.
- The second difference between the first two pairs of $y$-values ($x = 0$ and $x = 1$; $x = 1$ and $x = 2$) is $6a + 2b$.

$$6a + 2b = 0$$
$$1 + 2b = 0$$
$$2b = -1$$
$$b = -\frac{1}{2}$$

- The first difference between the $y$-values for $x = 0$ and $x = 1$ is equal to $a + b + c$.

$$a + b + c = 1$$
$$\frac{1}{6} - \frac{1}{2} + c = 1$$
$$c = \frac{8}{6} = \frac{4}{3}$$

**STEP 3** Write the cubic function rule with the values of $a$, $b$, $c$, and $d$:

$$f(x) = \frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{4}{3}x.$$

## YOU TRY IT! #3

For the data set below, determine if the relationship is a cubic function. If so, write a function relating the variables.

| $x$ | $y$ |
|---|---|
| 0 | −6 |
| 1 | $7\frac{1}{2}$ |
| 2 | 18 |
| 3 | $28\frac{1}{2}$ |
| 4 | 42 |
| 5 | $61\frac{1}{2}$ |
| 6 | 90 |

$y = \frac{1}{2}x^3 - 3x^2 + 16x - 6$

## PRACTICE/HOMEWORK

*For each table below, determine whether the set of data represents a linear, exponential, quadratic, or cubic function.*

**1.**

| $x$ | $f(x)$ |
|---|---|
| -1 | 0.2 |
| 0 | 1 |
| 1 | 5 |
| 2 | 25 |
| 3 | 125 |
| 4 | 625 |

**Exponential**

**2.**

| $x$ | $y$ |
|---|---|
| 0 | -1.25 |
| 1 | -1 |
| 2 | -0.75 |
| 3 | -0.5 |
| 4 | -0.25 |
| 5 | 0 |

**Linear**

**3.**

| $x$ | $f(x)$ |
|---|---|
| -1 | -5 |
| 0 | 0 |
| 1 | 5 |
| 2 | 40 |
| 3 | 135 |
| 4 | 320 |

**Cubic**

**4.**

| $x$ | $f(x)$ |
|---|---|
| -1 | -2 |
| 0 | -8 |
| 1 | -2 |
| 2 | 16 |
| 3 | 46 |
| 4 | 88 |

**Quadratic**

**5.**

| $x$ | $y$ |
|---|---|
| -1 | 8 |
| 0 | 5 |
| 1 | 10 |
| 2 | 29 |
| 3 | 68 |
| 4 | 133 |

**Cubic**

**6.**

| $x$ | $y$ |
|---|---|
| 1 | 40 |
| 2 | 38 |
| 3 | 36 |
| 4 | 34 |
| 5 | 32 |
| 6 | 30 |

**Linear**

## ADDITIONAL EXAMPLES

For the tables below, determine if the relationship is a cubic function. If so, write a function relating the variables.

**1.**

| $x$ | $y$ |
|---|---|
| 1 | 14.6 |
| 0 | 10 |
| 1 | 5.4 |
| 2 | 3.2 |
| 3 | 5.8 |
| 4 | 15.6 |

*Cubic,* $y = -\frac{2}{5}x^3 + 5x - 10$

**2.**

| $x$ | $y$ |
|---|---|
| -2 | -143.25 |
| -1 | -25.75 |
| 0 | -1.25 |
| 1 | 2.25 |
| 2 | 56.75 |
| 3 | 234.25 |

*Cubic,*
$y = 12x^3 - 10.5x^2 + 2x - 1.25$

**7.** Does the set of data shown below represent a cubic function? Justify your response.

| $x$ | $y$ |
|---|---|
| 0 | 0 |
| 1 | -4 |
| 2 | -28 |
| 3 | -76 |
| 4 | -148 |
| 5 | -244 |

***No, the data do not represent a cubic function; the differences in x are constant, but the third differences in y are not constant.***

*For questions 8 – 10, determine if the given relationship is a cubic function. If it is, write a function relating the variables.*

**8.**

| $x$ | $y$ |
|---|---|
| -1 | 8 |
| 0 | 5 |
| 1 | 6 |
| 2 | 11 |
| 3 | 20 |
| 4 | 33 |

***Not a cubic function***

**9.**

| $x$ | $f(x)$ |
|---|---|
| 0 | 0 |
| 1 | 0.5 |
| 2 | 4 |
| 3 | 13.5 |
| 4 | 32 |
| 5 | 62.5 |

***cubic; $f(x) = 0.5x^3$***

**10.**

| $x$ | $y$ |
|---|---|
| 0 | -7 |
| 1 | -5 |
| 2 | 9 |
| 3 | 47 |
| 4 | 121 |
| 5 | 243 |

***cubic; $y = 2x^3 - 7$***

*For questions 11 – 16, the data sets shown in the tables represent cubic functions. Write a cubic function for the values in the table.*

**11.**

| $x$ | $y$ |
|---|---|
| 0 | 0 |
| 1 | 0.25 |
| 2 | 2 |
| 3 | 6.75 |
| 4 | 16 |
| 5 | 31.25 |

***$y = 0.25x^3$***

**12.**

| $x$ | $f(x)$ |
|---|---|
| 0 | -5 |
| 1 | -4.8 |
| 2 | -3.4 |
| 3 | 0.4 |
| 4 | 7.8 |
| 5 | 20 |

***$f(x) = 0.2x^3 - 5$***

**13.**

| $x$ | $y$ |
|---|---|
| 0 | 1 |
| 1 | 9 |
| 2 | 57 |
| 3 | 181 |
| 4 | 417 |
| 5 | 801 |

***$y = 6x^3 + 2x^2 + 1$***

**14.**

| $x$ | $y$ |
|---|---|
| 0 | 0 |
| 1 | -4 |
| 2 | 0 |
| 3 | 18 |
| 4 | 56 |
| 5 | 120 |

***$y = x^3 + x^2 - 6x$***

**15.**

| $x$ | $f(x)$ |
|---|---|
| 0 | -1 |
| 1 | 0.3 |
| 2 | 7.4 |
| 3 | 25.1 |
| 4 | 58.2 |
| 5 | 111.5 |

***$f(x) = 0.8x^3 + 0.5x^2 - 1$***

**16.**

| $x$ | $y$ |
|---|---|
| 0 | 0 |
| 1 | -27 |
| 2 | -60 |
| 3 | -63 |
| 4 | 0 |
| 5 | 165 |

***$y = 6x^3 - 21x^2 - 12x$***

*Use the situation below to answer questions 17 – 18.*

### GEOMETRY

The volume of a set of rectangular prisms with a base length of $x$ inches, is shown below.

| LENGTH OF BASE, $x$ (INCHES) | VOLUME, $v(x)$ (CUBIC INCHES) |
|---|---|
| 0 | 0 |
| 1 | 1.5 |
| 2 | 12 |
| 3 | 40.5 |
| 4 | 96 |
| 5 | 187.5 |

**17.** Write the cubic function relating the length of the base to the volume.
***$v(x) = 1.5x^3$***

**18.** Use your function to predict the length of the base of the prism when the volume is 1500 cubic inches.
***The length of the base of the prism will be 10 inches.***

*Use the situation below to answer questions 19 – 20.*

### FINANCE

A local mail service charges different rates, based on the weight of the package being mailed. A sample of their prices is shown in the table below.

| WEIGHT OF PACKAGE, $w$ (POUNDS) | PRICE TO MAIL PACKAGE, $p$ ($) |
|---|---|
| 0 | 0 |
| 1 | 3.45 |
| 2 | 6.60 |
| 3 | 10.65 |
| 4 | 16.80 |
| 5 | 26.25 |

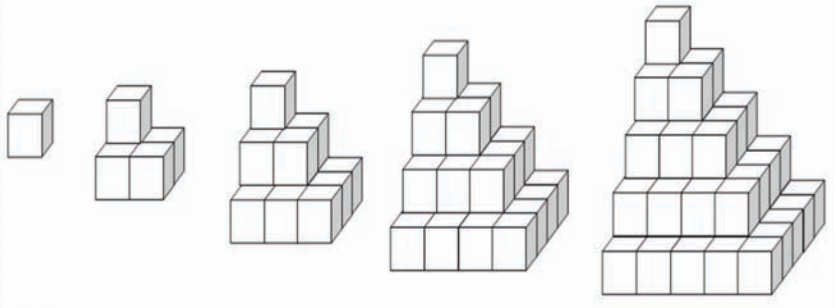
**19.** Write a cubic function to represent the given data.
***$p = 0.2w^3 - 0.75w^2 + 4w$***

**20.** Use your cubic function to determine the cost to mail a 6-pound package.
***The price to mail a 6-pound package would be $40.20.***

*Use the situation below to answer questions 21 – 23.*

## CRITICAL THINKING

Blocks were stacked to create the pattern below.



**21.** Relate the number of layers in a stack, $x$, to the total number of blocks, $y$, by completing the table below. The first few rows have been completed for you.

| NUMBER OF LAYERS, $x$ | TOTAL NUMBER OF BLOCKS, $p$ |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | 5 |
| 3 | 14 |
| 4 | ***30*** |
| 5 | ***55*** |

**22.** Write the function relating the variables in problem 21.
$y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x$

**23.** If the pattern continues, how many blocks would it take to create a 7-layer stack?
***It would take 140 blocks to create a 7-layer stack.***