Writing Exponential Functions

1.4

FOCUSING QUESTION What are the characteristics of an exponential function?

LEARNING OUTCOMES

- I can determine patterns that identify an exponential function from its related common ratios.
- I can classify a function as linear or exponential when I am given a table.
- I can determine the exponential function from a table using common ratios, including any restrictions on the domain and range.
- I can analyze patterns to connect the table to a function rule and communicate the exponential pattern as a function rule.

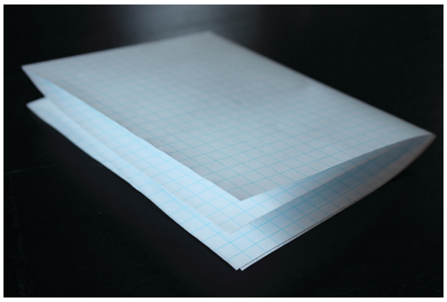
ENGAGE

Miranda shared a cookie recipe on social media with three friends. Each of Miranda's friends shared the cookie recipe with three of their friends. If this trend continues, how many people will receive a cookie recipe in the fifth round?

243 people.

EXPLORE

Begin with a sheet of paper. Fold it in half and record the number of layers of paper after the fold in a table like the one shown.



NUMBER OF FOLDS	NUMBER OF LAYERS
0	1
1	***2***
2	***4***
3	***8***
4	***16***
5	***32***
6	***64***

TEKS

AR.2A Determine the patterns that identify the relationship between a function and its common ratio or related finite differences as appropriate, including linear, quadratic, cubic, and exponential functions.

AR.2B Classify a function as linear, quadratic, cubic, and exponential when a function is represented tabularly using finite differences or common ratios as appropriate

AR.2C Determine the function that models a given table of related values using finite differences and its restricted domain and range.

MATHEMATICAL PROCESS SPOTLIGHT

AR.1F Analyze mathematical relationships to connect and communicate mathematical ideas.

ELPS

5F Write using a variety of grade-appropriate sentence lengths, patterns, and connecting words to combine phrases, clauses, and sentences in increasingly accurate ways as more English is acquired.

VOCABULARY

common ratio, base, exponent, multiplier, exponential relationship

MATERIALS

- 2 sheets of paper for each student

1. What is the difference between the numbers of folds in consecutive rows in the table?

1 fold

2. What is the difference between the numbers of layers in consecutive rows in the table?

NUMBER OF FOLDS	NUMBER OF LAYERS	
0	1	
1	*2*	$\Delta y = 2 - 1 = 1$
2	*4*	$\Delta y = 4 - 2 = 2$
3	*8*	$\Delta y = 8 - 4 = 4$
4	*16*	$\Delta y = 16 - 8 = 8$
5	*32*	$\Delta y = 32 - 16 = 16$
6	*64*	$\Delta y = 64 - 32 = 32$

3. Are the finite differences between the number of layers and the number of folds constant? How can you tell?

No, because while $\Delta x = 1$ and is constant for every successive row, Δy is not constant.

4. What patterns do you observe in the differences in the table?

See margin.

5. Is this a linear relationship? How do you know?

No, because the finite differences are not constant for the same Δx.

6. What is the ratio between successive numbers of layers?

$\frac{2}{1} = 2$ $\frac{4}{2} = 2$ $\frac{8}{4} = 2$ $\frac{16}{8} = 2$ $\frac{32}{16} = 2$ $\frac{64}{32} = 2$

7. How many layers would there be after the 7th fold? 10th fold?

7th fold: 128, 10th fold: 1024

8. What type of function best describes this relationship? Explain your reasoning.

See margin.

9. Write an equation that could be used to determine y, the number of layers, if you know x, the number of folds.

$y = 2^x$

4. *Possible response: The finite differences in the number of layers is the same as the number of layers shifted down one row. For each fold, the number of layers doubles.*

8. *Possible response: An exponential function best describes this relationship because each fold doubles the number of layers, so the number of layers is multiplied by the same number, 2, for every fold you make.*

QUESTIONING STRATEGIES

As students move through the second scenario, call attention to how they did the first scenario. Ask questions such as:

- How does your data for the area of the folded region compare to the data for the number of layers?
- Which of the two situations is an increasing function? Decreasing function?

Begin with a new sheet of paper.

10. What is area of the sheet of paper without any folds?

Possible response: If the paper is 8.5 × 11 inches, then the area is 93.5 square inches.

11. Fold the paper in half and record the area of the region showing after the fold in a table like the one shown. If necessary, round to the nearest tenth of a square inch.

Answers shown are for a regular sheet of 8.5 × 11 inch sheet of paper. If students use different size paper, then their answers will vary.

NUMBER OF FOLDS	AREA OF REGION
0	*93.5*
1	*46.8*
2	*23.4*
3	*11.7*
4	*5.8*
5	*2.9*
6	*1.5*

12. What is the difference between the numbers of folds in consecutive rows in the table?

1 fold

13. What is the difference between the areas of regions in consecutive rows in the table?

NUMBER OF FOLDS	AREA OF REGION
0	*93.5*
1	*46.8*
2	*23.4*
3	*11.7*
4	*5.8*
5	*2.9*
6	*1.5*

$\Delta y = 46.8 - 93.5 = -46.7$

$\Delta y = 23.4 - 46.8 = -23.4$

$\Delta y = 11.7 - 23.4 = -11.7$

$\Delta y = 5.8 - 11.7 = -5.9$

$\Delta y = 2.9 - 5.8 = -2.9$

$\Delta y = 1.5 - 2.9 = -1.4$

INSTRUCTIONAL HINTS

Suggest that students write measurements along the edges of the piece of paper each time they fold it.

Create a need for an equation rather than continuing the table. Ask students why an equation in helpful in both scenarios.

14. Are the finite differences between the area of the regions and the number of folds constant? How can you tell?

No, because while $\Delta x = 1$ and is constant for every successive row, Δy is not constant.

15. What patterns do you observe in the differences in the table?

See margin.

16. Is this a linear relationship? How do you know?

No, because the finite differences are not constant for the same Δx.

17. What is the ratio between successive areas of regions?

$\frac{46.8}{93.5} \approx 0.5 \quad \frac{23.4}{46.8} = 0.5 \quad \frac{11.7}{23.4} = 0.5 \quad \frac{5.8}{11.7} \approx 0.5 \quad \frac{2.9}{5.8} = 0.5 \quad \frac{1.5}{2.9} \approx 0.5$

18. What would be the area of the region present after the 7^{th} fold?

7^{th} fold: 0.8

19. What type of function best describes this relationship? Explain your reasoning.

See margin.

20. Write an equation that could be used to determine y, the area of the region, if you know x, the number of folds.

$y = 93.5\left(\frac{1}{2}\right)^x$

REFLECT

- **What do you notice about the successive ratios in each relationship?**
  See margin.
- **What relationship exists between the successive ratios in the dependent variable and the equations that you have written?**
  See margin.

EXPLAIN

When you found finite differences that were constant for the dependent variable, y, and the differences between values of the independent variable, x, were the same, the relationship was linear. But as you have seen, this is not true for every functional relationship.

If the finite differences are not constant, look at the ratios between successive rows in the table. If these ratios are constant, then the constant ratio is called a **common ratio** and the relationship is an **exponential function**. An exponential relationship is

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15. *Possible response: The finite differences in the are of region is the same as the number of layers shifted down one row. For each fold, the area of the new region is about half of the area of the previous region.*

19. *Possible response: An exponential function best describes this relationship because each fold generates a region that has half the area of the previous region.*

REFLECT ANSWERS:

When the difference in x-values is 1, the successive ratios are constant.

The successive ratio is the same as the base in the equation.

HIGHER-ORDER THINKING STRATEGY

Use a graphic organizer such as a Venn diagram to help students compare and contrast linear relationships and exponential relationships. The Venn diagram could be constructed in an interactive math notebook or math journal and students could add more information as they learn more about linear and exponential relationships in this chapter.

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one in which there is repeated multiplication. For example, a geometric sequence has a constant multiplier, so the sequence can be represented as an exponential function with a domain of whole numbers.

In an exponential function, if the base is greater than 1, then the function represents **exponential growth**. If the base is between 0 and 1, then the function represents **exponential decay**.

Let's look more closely at an exponential function. The table below shows the relationship between x and $f(x)$. In an exponential function, $f(x) = ab^x$, b represents the base of the exponential function, which is also a common ratio or constant multiplier. The parameter a represents an initial value or y-intercept.

x	$y = f(x)$
0	a
1	ab
2	$ab(b)$
3	$ab(b)(b)$
4	$ab(b)(b)(b)$
5	$ab(b)(b)(b)(b)$

$\Delta x = 1 - 0 = 1$; $\Delta y = ab - a = a(b - 1)$

$\Delta x = 2 - 1 = 1$; $\Delta y = ab(b) - ab = ab(b - 1)$

$\Delta x = 3 - 2 = 1$; $\Delta y = ab(b)(b) - ab(b) = ab^2(b - 1)$

$\Delta x = 4 - 3 = 1$; $\Delta y = ab(b)(b)(b) - ab(b)(b) = ab^3(b - 1)$

$\Delta x = 5 - 4 = 1$; $\Delta y = ab(b)(b)(b)(b) - ab(b)(b)(b) = ab^4(b - 1)$

Unlike a linear function, the finite differences for an exponential function, $f(x) = ab^x$, are not constant. Instead, the multiplicative pattern that is present in the original data repeats in the finite differences.

Instead of looking at the finite differences, for an exponential function, take a closer look at the successive ratios.

x	$y = f(x)$
0	a
1	ab
2	$ab(b)$
3	$ab(b)(b)$
4	$ab(b)(b)(b)$
5	$ab(b)(b)(b)(b)$

$\Delta x = 1 - 0 = 1$; $\frac{y_n}{y_{n-1}} = \frac{ab}{a} = b$

$\Delta x = 2 - 1 = 1$; $\frac{y_n}{y_{n-1}} = \frac{a(b)(b)}{a(b)} = b$

$\Delta x = 3 - 2 = 1$; $\frac{y_n}{y_{n-1}} = \frac{a(b)(b)(b)}{a(b)(b)} = b$

$\Delta x = 4 - 3 = 1$; $\frac{y_n}{y_{n-1}} = \frac{a(b)(b)(b)(b)}{a(b)(b)(b)} = b$

$\Delta x = 5 - 4 = 1$; $\frac{y_n}{y_{n-1}} = \frac{a(b)(b)(b)(b)(b)}{a(b)(b)(b)(b)} = b$

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Notice that in the table, $\Delta x = 1$. Knowing that, the common ratio for successive rows is equivalent to b, which is the base of the exponential relationship.

COMMON RATIOS AND EXPONENTIAL FUNCTIONS

In an exponential function, the ratios between successive y-values, $\frac{y_n}{y_{n-1}}$, are constant if the differences between successive x-values, Δx, are also constant.

If the ratios of successive values of the dependent variable in a table of values are constant, then the values represent an exponential function.

You can also use the common ratio to write an exponential function describing the relationship between the independent and dependent variables.

EXAMPLE 1

Does the set of data shown below represent an exponential function? Justify your answer.

x	y
0	1
1	5
2	25
3	125
4	625

INTEGRATE TECHNOLOGY

Use technology such as a graphing calculator or spreadsheet app on a display screen to show students how geometric sequences are related to exponential functions. If the exponential function's domain is restricted to whole numbers, then the resulting points represent a geometric sequence.

STEP 1 Determine the finite differences between successive x-values and the ratios between successive y-values.

	x	y	
	0	1	
$\Delta x = 1 - 0 = 1$			$\frac{y_n}{y_{n-1}} = \frac{5}{1} = 5$
	1	5	
$\Delta x = 2 - 1 = 1$			$\frac{y_n}{y_{n-1}} = \frac{25}{5} = 5$
	2	25	
$\Delta x = 3 - 2 = 1$			$\frac{y_n}{y_{n-1}} = \frac{125}{25} = 5$
	3	125	
$\Delta x = 4 - 3 = 1$			$\frac{y_n}{y_{n-1}} = \frac{625}{125} = 5$
	4	625	

STEP 2 Determine whether or not the differences between successive x-values and ratios between successive y-values are constant.

The differences between successive values of x, Δx, are all 1, so they are constant.

The ratios between successive values of y, $\frac{y_n}{y_{n-1}}$, are all 5, so they are constant.

$$\frac{\frac{y_n}{y_{n-1}}}{\Delta x} = \frac{5}{1} = 5 \text{ for all pairs of } \Delta x \text{ and } \frac{y_n}{y_{n-1}}.$$

STEP 3 Determine whether or not the set of data represents an exponential function.

Yes, the set of data represents an exponential function because the differences between successive x-values and the ratios between successive y-values are constant.

ADDITIONAL EXAMPLES

Determine if each of the data sets below represent an exponential function. Justify your answer.

1.

x	0	1	2	3	4
y	2	6	18	54	162

Exponential

2.

x	-1	0	1	2	3
y	8	4	2	1	0.5

Exponential

3.

x	3	6	9	12	15
y	12	17	22	27	32

Not Exponential

Note: Pay special attention to students' justification for the third additional example. Be mindful that some students may say "no, because the differences between successive values of x are not 1." The differences are constant, but the ratio between successive values of y are not constant.

YOU TRY IT! #1 ANSWER:

Yes, the set of data represents an exponential function, because the finite differences in x and the successive ratios in y are both constant.

YOU TRY IT! #1

Does the set of data shown below represent an exponential function? Justify your answer.

x	y
0	1.2
1	1.44
2	1.728
3	2.0736

See margin.

EXAMPLE 2

Does the set of data shown below represent an exponential function? Justify your answer.

x	y
0	8
1	4
2	2
3	0.8
4	0.4

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STEP 1 Determine the finite differences between successive x-values and ratios between successive y-values.

x	y
0	8
1	4
2	2
3	0.8
4	0.4

$\Delta x = 1 - 0 = 1$

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\frac{y_n}{y_{n-1}} = \frac{4}{8} = \frac{1}{2}$

$\frac{y_n}{y_{n-1}} = \frac{2}{4} = \frac{1}{2}$

$\frac{y_n}{y_{n-1}} = \frac{0.8}{2} = \frac{2}{5}$

$\frac{y_n}{y_{n-1}} = \frac{0.4}{0.8} = \frac{1}{2}$

STEP 2 Determine whether or not the differences are constant.

The differences in x, Δx, are all 1, so they are constant.

The ratios of successive values of y, $\frac{y_n}{y_{n-1}}$, are not all the same, so they are not constant.

STEP 3 Determine whether or not the set of data represents an exponential function.

No, the set of data does not represent an exponential function because even though the differences in successive values of x are constant, the ratios between successive values of y are not constant.

YOU TRY IT! #2

Does the set of data shown below represent an exponential function? Justify your answer.

x	y
0	9.2
1	18.4
2	46
3	92
4	230

See margin.

YOU TRY IT! #2 ANSWER:

No, the set of data does not represent an exponential function, because although the first differences in x are constant, the ratios between successive values of y are not constant.

ADDITIONAL EXAMPLES

Determine if each of the data sets below represent an exponential function. If so, write a function relating the variables. Justify your answer.

1.

x	0	1	2	3	4
y	-1	-2	-4	-8	-16

Exponential, $y = -2^x$

2.

x	18	15	12	9
y	24	22	20	18

Not Exponential

3.

x	2	3	5	8	12
y	$-\frac{2}{3}$	1	-1.5	2.25	-3.375

Not Exponential

4.

x	1	2	3	4
y	4.5	6.75	10.125	15.1875

Exponential, $y = 3(1.5)^x$

EXAMPLE 3

For the data set below, determine if the relationship is an exponential function. If so, write a function relating the variables.

x	y
1	4
2	10
3	25
4	62.5
5	156.25

STEP 1 **Determine the finite differences between successive x-values and the ratios between successive y-values.**

Δx	x	y	$\frac{y_n}{y_{n-1}}$
	1	4	
$\Delta x = 2 - 1 = 1$	2	10	$\frac{y_n}{y_{n-1}} = \frac{10}{4} = 2.5$
$\Delta x = 3 - 2 = 1$	3	25	$\frac{y_n}{y_{n-1}} = \frac{25}{10} = 2.5$
$\Delta x = 4 - 3 = 1$	4	62.5	$\frac{y_n}{y_{n-1}} = \frac{62.5}{25} = 2.5$
$\Delta x = 5 - 4 = 1$	5	156.25	$\frac{y_n}{y_{n-1}} = \frac{156.25}{62.5} = 2.5$

STEP 2 **Determine whether or not the relationship is an exponential function.**

The differences in x, Δx, are all 1, so they are constant.

The ratios between successive values of y, $\frac{y_n}{y_{n-1}}$, are all 2.5, so they are constant.

Since the first differences in x and the successive ratios in y are all constant, the relationship is an exponential function.

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STEP 3 **Determine the y-intercept of the exponential function.**

Work backwards from $x = 1$ and $y = 4$.

x	y
0	a
1	4
2	10

$1 - 0 = 1$, $2 - 1 = 1$; $\frac{4}{a} = 2.5$, $\frac{10}{4} = 2.5$

$$\frac{4}{a} = 2.5$$
$$a\left(\frac{4}{a}\right) = 2.5(a)$$
$$4 = 2.5a$$
$$\frac{4}{2.5} = \frac{2.5a}{2.5}$$
$$1.6 = a$$

The y-intercept is (0, 1.6).

STEP 4 **Use the y-coordinate of the y-intercept and the common ratio to write the exponential function.**

$y = 1.6(2.5)^x$

YOU TRY IT! #3

For the data set below, determine if the relationship is an exponential function. If so, write a function relating the variables.

x	y
1	14
2	98
3	686
4	4802

Answer: $y = 2(7)^x$

ADDITIONAL EXAMPLES

After completing the additional examples on pg. 46, draw students' attention back to examples 2 and 3. Based on prior lessons, ask if they can write a function rule for either of those sets of data?

Yes, Additional Example #2 is linear. The function rule is $y = \frac{2}{3}x - 12$.

QUESTIONING STRATEGIES

Assist students in processing their learning by having students write the steps in their own words.

- How would you explain to a friend how you know if a set of data represents an exponential function?
- Explain how to write an exponential function based on a table of values to someone who has never seen this math lesson.

EXAMPLE 4

For the data set below, determine if the relationship is an exponential function. If so, write a function relating the variables.

x	y
1	1000
2	400
3	160
4	64
5	25.6

STEP 1 Determine the first differences between successive x-values and the ratios between successive y-values.

x	y
1	1000
2	400
3	160
4	64
5	25.6

$\Delta x = 2 - 1 = 1$

$\Delta x = 3 - 2 = 1$

$\Delta x = 4 - 3 = 1$

$\Delta x = 5 - 4 = 1$

$\frac{y_n}{y_{n-1}} = \frac{400}{1000} = 0.4$

$\frac{y_n}{y_{n-1}} = \frac{160}{400} = 0.4$

$\frac{y_n}{y_{n-1}} = \frac{64}{160} = 0.4$

$\frac{y_n}{y_{n-1}} = \frac{25.6}{64} = 0.4$

STEP 2 Determine whether or not the relationship is an exponential function.

The differences in x, Δx, are all 1, so they are constant.

The ratios between successive values of y, $\frac{y_n}{y_{n-1}}$, are all 0.4, so they are constant.

Since the first differences in x and the successive ratios in y are all constant, the relationship is an exponential function.

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STEP 3 **Determine the y-intercept of the exponential function.**

Work backwards from $x = 1$ and $y = 1000$.

x	y
0	a
1	1000
2	400

$1 - 0 = 1$; $\frac{1000}{a} = 0.4$

$2 - 1 = 1$; $\frac{400}{1000} = 0.4$

$$\frac{1000}{a} = 0.4$$
$$a\left(\frac{1000}{a}\right) = 0.4(a)$$
$$1000 = 0.4a$$
$$\frac{1000}{0.4} = \frac{0.4a}{0.4}$$
$$2500 = a$$

The y-intercept is (0, 2500).

STEP 4 **Use the y-coordinate of the y-intercept and the common ratio to write the exponential function.**

$y = 2500(0.4)^x$

YOU TRY IT! #4

For the data set below, determine if the relationship is an exponential function. If so, write a function relating the variables.

x	y
1	81
2	27
3	9
4	3
5	1

Answer: $y = 243\left(\frac{1}{3}\right)^x$

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PRACTICE/HOMEWORK

For questions 1-4 use finite differences to determine if each table represents an exponential function. Use real objects such as pennies or beans to create 4 stacks, one to represent the y-value for each x-value, and use the stacks to help you determine whether or not the data represents an exponential function.

1.

x	y
0	2
1	6
2	18
3	54

Yes

2.

x	y
0	3
1	4
2	7
3	12

No

3.

x	y
0	0
1	1
2	8
3	27

No

4.

x	y
0	3
1	6
2	12
3	24

Yes

For questions 5-8 identify if each table represents an exponential function or not. If the table represents an exponential function, identify the common ratio.

5.

x	y
1	2
2	4
3	6
4	8

Exponential Function? ***No***
Common Ratio: ***None***

6.

x	y
1	2
2	4
3	8
4	16

Exponential Function? ***Yes***
Common Ratio: ***2***

7.

x	y
1	3
2	4.5
3	6.75
4	10.125

Exponential Function? ***Yes***
Common Ratio: ***1.5***

8.

x	y
1	4
2	1
3	0.25
4	0.0625

Exponential Function? ***Yes***
Common Ratio: ***0.25***

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For questions 9-12 use the situation below.

CRITICAL THINKING

A sheet of paper is 0.1 mm thick. When the paper is folded in half, the total thickness of the layers of paper is 0.2 mm. When the paper is folded in half again, the total thickness of the layers of paper is 0.4 mm.

9. Complete the table below to represent the situation.

NUMBER OF FOLDS x	TOTAL THICKNESS OF LAYERS y
0	0.1
1	0.2
2	***0.4***
3	***0.8***
4	***1.6***

10. Does the situation represent a linear function or an exponential function? Justify your answer.
See margin.

11. Which of the following represents the function that models this situation?

A. $y = x + 0.1$
B. $y = 2 \bullet 0.1^x$
C. $y = 0.1 \bullet 2^x$
D. $y = 2^x + 0.1$

12. Which of the following statements are true about the situation?

- $\Delta x = 1$
- The situation is an example of exponential decay.
- The function is increasing.
- The common ratio is 2.
- The y-intercept is (0, 0.1).
- The function is linear.
- The function is decreasing.
- $\Delta y = 0.1$
- The common ratio is 0.2.
- The situation is an example of exponential growth.

*For questions 13-18 identify if each table represents an exponential function or not. If the table **represents** an exponential function, write the function relating the variables.*

13.

x	y
0	0
1	4
2	32
3	108

Exponential Function? ***No***
Function: ***N/A***

14.

x	y
0	40
1	8
2	1.6
3	0.32

Exponential Function? ***Yes***
Function: ***$y = 40(0.2)^x$***

10. *Exponential function*

While Δx is constant, Δy is not constant. The situation does have a common ratio of 2.

15.

x	y
0	50
1	25
2	12.5
3	6.25

Exponential Function? ***Yes***
Function: $y = 50(0.5)^x$

16.

x	y
1	300
2	150
3	100
4	75

Exponential Function? ***No***
Function: ***N/A***

17.

x	y
1	4500
2	6750
3	10,125
4	15,187.5

Exponential Function? ***Yes***
Function: $y = 3{,}000(1.5)^x$

18.

x	y
1	14
2	56
3	224
4	896

Exponential Function? ***Yes***
Function: $y = 3.5(4)^x$

For questions 19-20 use the situation below.

CRITICAL THINKING

A sheet of paper has an area of 100 square inches. When the paper is cut in half, the area of one piece is 50 square inches. When that piece is cut in half, the area of one piece is 25 square inches.

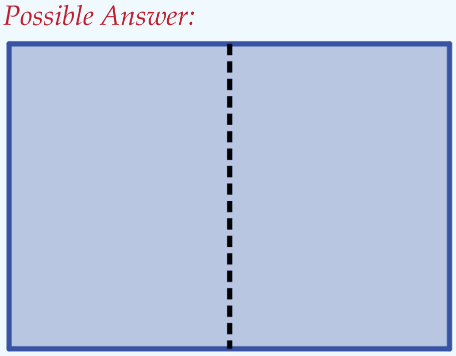
NUMBER OF CUTS x	AREA OF ONE PIECE y
0	100
1	50
2	25

19. What would be the area of one piece after 5 cuts?
3.125 square inches

20. Write the function relating the variables.
$y = 100(0.5)^x$

21. Draw a diagram of the paper and how it is cut in half. Use the diagram to interpret the values of a and b in your exponential function. Communicate your mathematical reasoning and its implications using the diagram.
See margin.

21. *Possible Answer:*



The value of a, 100, is the area of the original sheet of paper. The value of b, 0.5, represents cutting the paper in half each time. For each successive cut, the area of paper is reduced again by a factor of 0.5.

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