Modeling with Linear Functions

1.3

FOCUSING QUESTION How can you use finite differences to construct a linear model for a data set?

LEARNING OUTCOMES

- I can use finite differences to write a linear function that describes a data set.
- I can apply mathematics to problems that I see in everyday life, in society, and in the workplace.

ENGAGE

Mariette, a dendrochronologist, observed that some tree stumps have rings that are close together while other tree stumps have rings that are farther apart. Why does a tree stump have rings? What might cause the rings to be closer together or farther apart?

See margin.

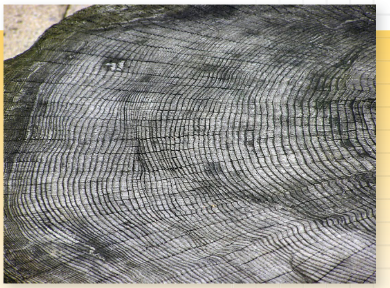

Image credit: Adrian Pingstone, Tree ring, Wikimedia Commons

EXPLORE

Each year during the growing season, trees grow larger by adding another layer of cells just beneath the bark. This layer is called a tree ring. Because a tree ring is added each year, scientists can determine the age of a tree by counting the number of tree rings that are present.

However, not all tree rings have the same width. Trees grow more when there is plenty of rain and the soil is fertile. Scientists can draw conclusions about temperature and rainfall for a particular year based on the width of the tree ring for that year.

Mariette measured the width of tree rings from a core sample she took from a post oak tree in the Brazos River valley of central Texas. From the tree ring width, she calculated the radius of the tree. The table below shows her results.

YEAR	2000	2001	2002	2003	2004	2005	2006
YEAR NUMBER	0	1	2	3	4	5	6
RADIUS (CM)	2.5	3.1	3.5	3.9	4.4	4.9	5.5

TEKS

AR.2D Determine a function that models real-world data and mathematical contexts using finite differences such as the age of a tree and its circumference, figurative numbers, average velocity, and average acceleration.

MATHEMATICAL PROCESS SPOTLIGHT

AR.1A Apply mathematics to problems arising in everyday life, society, and the workplace.

ELPS

5B Write using newly acquired basic vocabulary and content-based grade-level vocabulary.

VOCABULARY

linear function, finite differences

MATERIALS

- graphing calculator, spreadsheet, or graphic application

ENGAGE ANSWER:

Possible answer: The rings are formed because, as the tree grows, each year it adds a new layer of cells under the bark. The amount of rain might cause the rings to be closer together or farther apart.

TECHNOLOGY INTEGRATION

When modeling with real-world data, a scatterplot of the data set and the function model helps students visualize the relationship between the two variables. Scatterplots can be made using graphing calculators, spreadsheets, or graphing apps.

ELL STRATEGY

Writing with newly acquired vocabulary (ELPS: c5B) helps English language learners internalize the vocabulary terms that they have recently learned. Using vocabulary from current and past learning experiences (e.g., linear function, finite differences) to explain their thinking and mathematical reasoning reinforces how these terms are consistent through a variety of settings and contexts.

REFLECT ANSWERS:

Use an average value of the first differences as the slope of a linear function model.

Use your linear function model to write an equation where the linear function is equal to a particular value. Then, solve for x.

1. Calculate the finite differences between the year number and the radius.
   See margin
2. Are the first differences in the radius constant? Explain how you know.
   The first differences in radius are not exactly constant but are very close in value.
3. What is the average finite difference in radius?
   0.5 cm
4. Use the information from the table to write a function rule that models the data.
   f(x) = 2.5 + 0.5x, where x represents the year number, or number of years since 2000.
5. What do the slope and y-intercept from your function rule mean in the context of this situation?
   See margin
6. Use your model to predict the radius of the tree in 2015.
   f(15) = 2.5 + 0.5(15) = 10 centimeters
7. In what year will the radius of the tree be 12.5 centimeters?
   f(x) = 2.5 + 0.5x = 12.5, x = 20, so the year will be 2020
8. What would the circumference of the tree be in 2015?
   C = 2πr = 2π(10) = 20π ≈ 62.8 centimeters
9. Make a scatterplot of your data set and graph the function model over the scatterplot. How well would you say the function model predicts the actual values in the data set? Explain your reasoning.
   See margin

REFLECT

- **How can you determine a linear function model for a data set if the first differences are not exactly the same, but are almost constant?**
  See margin.
- **Once you have your linear function model, how can you use the model to determine a value of the independent variable that generates a particular value of the dependent variable?**
  See margin.

1.

+1 +1 +1 +1 +1 +1

YEAR	2000	2001	2002	2003	2004	2005	2006
YEAR NUMBER	0	1	2	3	4	5	6
RADIUS (CM)	2.5	3.1	3.5	3.9	4.4	4.9	5.5

+0.6 +0.4 +0.4 +0.5 +0.5 +0.6

5. *The slope, 0.5 centimeters per year, represents the growth rate of the tree, or the number of centimeters that the radius of the tree grows each year.*

The y-intercept, 2.5 centimeters, represents the radius of the tree in the year 2000, when the data were first collected.

9. *The function model connects several of the data values and is very close to the remaining data values. The function model appears to closely predict the actual data values.*

See page 27.

26 CHAPTER 1: ALGEBRAIC PATTERNS

EXPLAIN

A linear function model can be used to represent sets of mathematical and real-world data. Dendrochronologists use core samples, or cylinders that are about 5 millimeters in diameter that are drilled into and extracted from the tree to measure the width of tree rings. Once they have their model, they can use different measures, such as circumference of a tree, to calculate the age of the tree.

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You can use Mariette's data to generate a model relating the circumference of a tree to the age of the tree. For trees that were planted in 2000, the table shows the growth rate.

YEAR	2000	2001	2002	2003	2004	2005	2006
YEAR NUMBER	0	1	2	3	4	5	6
RADIUS (CM)	2.5	3.1	3.5	3.9	4.4	4.9	5.5

Add a new row to the table to calculate the circumference. Recall that circumference can be calculated using the formula $C = 2\pi r$, where r represents the radius of the circle and C represents the circumference of the circle. Round the circumference to the nearest tenth if necessary.

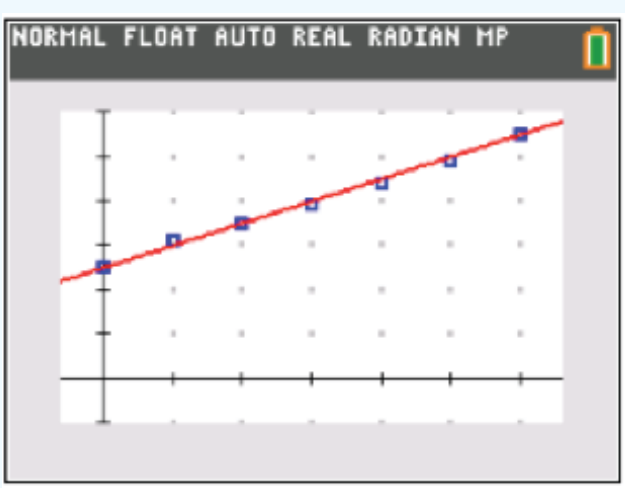
YEAR	2000	2001	2002	2003	2004	2005	2006
YEAR NUMBER	0	1	2	3	4	5	6
RADIUS (CM)	2.5	3.1	3.5	3.9	4.4	4.9	5.5
CIRCUMFERENCE (CM)	15.7	19.5	22.0	24.5	27.6	30.8	34.5

Use the rows for year number and circumference to calculate the first finite differences.

+1 +1 +1 +1 +1 +1

YEAR NUMBER	0	1	2	3	4	5	6
CIRCUMFERENCE (CM)	15.7	19.5	22.0	24.5	27.6	30.8	34.5

+3.8 +2.5 +2.5 +3.1 +3.2 +3.7



These first finite differences are not equal, but are all close to +3. Calculate the average finite difference, and use that to determine the slope of the linear function model.

$$\Delta y = \frac{3.8 + 2.5 + 2.5 + 3.1 + 3.2 + 3.7}{6} \approx 3.13$$

$$\Delta x = 1$$

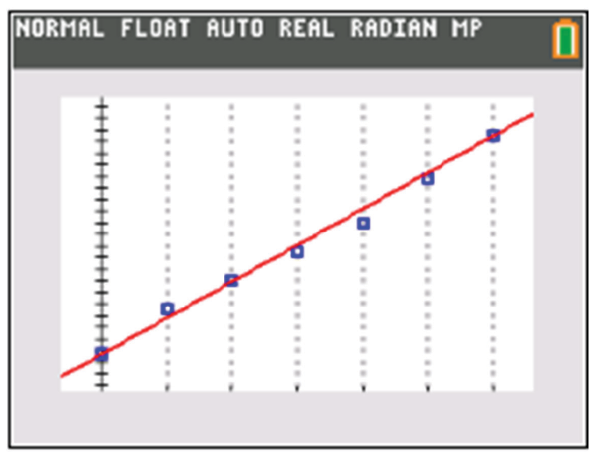
$$\frac{\Delta y}{\Delta x} = \frac{3.13}{1} = 3.13$$

Using the slope and y-intercept, you can write the function model, $f(x) = 15.7 + 3.13x$. Once you have a function model, you can use that model to make predictions.

MODELING WITH LINEAR FUNCTIONS

Real-world data rarely follows exact patterns, but you can use patterns in data to look for trends. If the data increases or decreases at about the same rate, then a linear function model may be appropriate for the data set.

You can also use a scatterplot and a graph to show how the values in the data set are related to the function model. The graph of the function model could also be useful in making predictions from the model.



$f(x) = 15.7 + 3.13x$

28 CHAPTER 1: ALGEBRAIC PATTERNS

EXAMPLE 1

A student takes small steps away from a motion detector at an approximately constant rate. The time, in seconds, for which the student walks and the distance, in meters, the student walks are recorded in the table.

TIME (S)	0	1	2	3	4
DISTANCE (M)	0.25	0.85	1.55	2.2	2.75

Generate a linear function model for this situation. Based on your model, how far away will the student be from the motion detector after 10 seconds?

STEP 1 Calculate the finite differences in the table.

+1 +1 +1 +1

TIME (S)	0	1	2	3	4
DISTANCE (M)	0.25	0.85	1.55	2.2	2.75

+0.6 +0.7 +0.65 +0.55

STEP 2 Calculate the average finite difference in the table and use this to determine the slope, or average velocity, for a linear function model.

$$\frac{0.6 + 0.7 + 0.65 + 0.55}{4} = \frac{2.5}{4} = 0.625$$

STEP 3 Use the slope and y-intercept to write a linear function model.

$$y = 0.625x + 0.25$$

STEP 4 Use your linear function model to make a prediction.

$$y = 0.625(10) + 0.25$$
$$y = 6.25 + 0.25$$
$$y = 6.5$$

According to the linear function model, the student will be 6.5 meters away from the motion detector after 10 seconds.

ADDITIONAL EXAMPLE

Generate a linear function model for the situation below and answer the questions.

A football player gets the ball at the 50 yard line and makes a clean break running for the end zone at the 0 yard line. His distance, in yards, over the time of his run, in seconds, is recorded in the table. If he does not get tackled, how many seconds will it take him to reach the end zone? Why would the finite differences not be constant in this scenario?

TIME (S)	0	1	2	3	4
DISTANCE (YD)	0	8.2	16.9	25.4	34

He will reach the end zone in between 5 and 6 seconds.

YOU TRY IT! #1 ANSWER:

$y = 8.325x + 0.175$; according to the linear function model, Tracy will run the half-marathon in approximately 109 minutes.

YOU TRY IT! #1

Tracy, a long distance runner, times herself as she runs a half-marathon, which is 13.1 miles long. The distance is measured in miles and the time is measured in minutes.

DISTANCE (MI)	1	2	3	4	5
TIME (MIN)	8.5	16.2	24.6	33.1	41.8

Generate a linear function model for this situation. Based on your model, how long to the nearest minute will it take Tracy to run the half-marathon?

See margin

EXAMPLE 2

Engineers conducting experiments in accident reconstruction want to see how long it would take a vehicle on a highway to coast to a stop if the brakes were inoperable. The first few seconds of the experiment are recorded in the table below. Time is measured in seconds and speed is measured in miles per hour.

TIME (S)	1	2	3	4	5
SPEED (MPH)	65	62	58	56	53

Generate a linear function model for this situation. Based on your model, when will the car come to a stop, to the nearest second?

STEP 1 **Calculate the first finite differences in the table.**

		+1		+1		+1		+1	
TIME (S)	1		2		3		4		5
SPEED (MPH)	65		62		58		56		53
		-3		-4		-2		-3	

30 CHAPTER 1: ALGEBRAIC PATTERNS

STEP 2 Calculate the average first finite difference in the table and use this to determine the slope, or average acceleration, for a linear function model.

$$\frac{-3 + (-4) + (-2) + (-3)}{4} = \frac{-12}{4} = -3$$

STEP 3 Use the average acceleration and first finite differences in the x-values to determine the y-intercept of a linear function model.

+1

TIME	0	1
SPEED	b	65

$65 - b = -3$

$$\begin{aligned} 65 - b &= -3 \\ 65 - b + 3 &= -3 + 3 \\ 68 - b &= 0 \\ 68 - b + b &= 0 + b \\ 68 &= b \end{aligned}$$

STEP 4 Write a linear function model.

$y = -3x + 68$

STEP 5 Use the linear function model to make a prediction.

$$\begin{aligned} 0 &= -3x + 68 \\ 0 - 68 &= -3x + 68 - 68 \\ -68 &= -3x \\ (-68) \div (-3) &= (-3x) \div (-3) \\ 22.667 &\approx x \end{aligned}$$

According to the linear function model, the car will come to a stop after approximately 23 seconds.

ADDITIONAL EXAMPLE

Mrs. Norris started the school year with a large supply of pencils for her students to borrow and use. Each month she noticed that her supply had dwindled. After one month she recorded how many pencils remained. Each month after she counted and recorded again. The data for the first 4 months of the school year are shown in the table below.

TIME (MONTHS)	1	2	3	4
PENCILS	195	185	173	159

Based on the model, how many months will it take for Mrs. Norris to drop below 100 pencils? Will she have enough pencils to last the school year? How many pencils did Mrs. Norris have at the start of the school year?

It would take about 9 months for Mrs. Norris' pencil supply to drop below 100 pencils. She will have enough pencils to last the school year. She started the year with about 207 pencils.

YOU TRY IT! #2 ANSWER:

$y = -189.75x + 1550$; the checking account Caleb's parents set up will have a balance below \$100 after 8 months.

YOU TRY IT! #2

Caleb's parents set up a checking account for him before college so that he will be able to pay the utilities for his apartment. Caleb keeps track of his spending in the table below. Time represents the number of months he has been in his apartment and the checking account balance is measured in dollars.

TIME (MONTHS)	0	1	2	3	4
BALANCE	\$1550	\$1355	\$1170	\$978	\$791

Generate a linear function model for this situation. Based on your model, when will the checking account balance dip below \$100?

See margin

PRACTICE/HOMEWORK

For the following sets of data, calculate the average finite difference, and use that to determine the slope of a linear function that could model the data.

1.

x	y
1	15.3
2	25.3
3	35.2
4	45.4
5	55.2
6	65.3

10

2.

x	0	1	2	3	4	5	6
y	50	47.2	44.3	41.5	38.5	35.6	32.5

-2.917

3.

x	1	2	3	4	5
y	14.25	14.05	15.35	16	16.55

0.575

32 CHAPTER 1: ALGEBRAIC PATTERNS

For problems 4 – 6, determine a linear function to model the situation.

FINANCE

4. Madeleine has a gift card to her favorite coffee shop. The table below shows how much is remaining on the gift card after each purchase at the coffee shop.

PURCHASES	0	1	2	3	4	5
BALANCE (DOLLARS)	40	35.68	31.22	26.97	22.65	18.40

y = 40 – 4.32x or y = -4.32x + 40, where x represents the number of purchases and y represents the balance remaining on the gift card.

SCIENCE

5. Gus records the mileage on his car, so he can determine his average mileage per month. Below are some of his collected data.

TIME (MONTHS)	1	2	3	4	5	6
MILEAGE (MILES)	11,540	12,482	13,570	14,670	15,682	16,757

y = 10,496.6 + 1043.4x or y = 1043.4x + 10,496.6, where x represents time and y represents mileage.

FINANCE

6. David is purchasing apps for his cell phone. The table below shows how his total cost changes with each app that he selects.

NUMBER OF APPS PURCHASED	1	2	3	4	5
COST (DOLLARS)	1.25	2.50	3.75	5.00	6.25

y = 1.25x, where x represents the number of apps purchased and y represents total cost.

Use the following situation to answer problems 7 – 10.

SCIENCE

Charlie is measuring his little brother's height throughout the year to see how much he grows. The table below shows how his height changes during the first 5 months.

TIME (MONTHS)	0	1	2	3	4	5
HEIGHT (INCHES)	54	54.20	54.45	54.85	55.10	55.25

7. Write a function rule to model the situation.
f(x) = 54 + 0.25x or f(x) = 0.25x + 54; where x represents the number of months.

8. What do the slope and *y*-intercept from your function rule mean in the context of this situation?
See margin

9. Use your model to predict the height of Charlie's brother after a year.
f(12) = 54 + 0.25(12) = 57 inches

10. In what month will his height be approximately 56 inches?
f(x) = 54 + 0.25x = 56, x = 8 months

Use the following situation to answer problems 11 – 13.

CRITICAL THINKING

Jeff noticed that the nutrition information on his box of cereal states that there are 14 servings in the cereal box. He decided to put their claim to the test. He recorded the weight of the remaining cereal after each serving, as shown in the table below.

NUMBER OF SERVINGS	0	1	2	3	4	5
WEIGHT OF REMAINING CEREAL (OUNCES)	14	12.7	11.6	10.7	9.5	8.4

11. Write a function rule that models the situation.
f(x) = 14 – 1.12x or f(x) = -1.12x + 14, where x represents the number of cereal servings

12. What do the slope and *y*-intercept from your function rule mean in the context of this situation?
See margin

13. Was Jeff able to confirm the claim on the cereal box by eating 14 servings? Explain your answer.
See margin

8. *The slope, 0.25 inches per month, represents the growth rate of Charlie's brother (how much his height increases each month). The y-intercept, 54 inches, represents the height of Charlie's brother when he started tracking his height.*

12. *The slope represents a decrease of 1.12 ounces per serving (each serving is 1.12 ounces). The y-intercept, 14 ounces, represents the starting weight of the box of cereal.*

13. *No.*

f(x) = 14 – 1.12x = 0; x = 12.5. He will finish the cereal at about 12.5 servings, instead of 14 servings. However, he may have used more than the suggested number of ounces per serving, so his results are not conclusive.

34 CHAPTER 1: ALGEBRAIC PATTERNS

Use the following situation to answer problems 14 – 17.

SCIENCE

Bob is tracking a hurricane moving toward the coast of Florida. The table below shows its distance from land over time.

TIME (HOURS)	0	1	2	3	4
DISTANCE (MILES)	704	684	663.7	644.2	624.4

14. Write a function rule that models the situation.
See margin

15. What do the slope and *y*-intercept from your function rule mean in the context of this situation?
See margin

16. About how far will the hurricane be from land after 24 hours?
f(24) = 704 – 19.9(24) = 226.4 miles from land

17. Approximately when will the hurricane make landfall?
f(x) = 704 – 19.9x = 0; x = 35.38; in a little over 35 hours

Use the following situation to answer problems 18 – 19.

CRITICAL THINKING

Maddie is running a 10-K (10 kilometer) race. She wears an electronic chip that tracks her progress throughout the race. She runs at a fairly steady pace throughout the race, as shown in her chip data below.

DISTANCE (KILOMETERS)	0	1	2	3	4	5
TIME (MINUTES)	0	4.1	7.9	11.8	15.5	19

18. Write a function rule that models the situation.
f(x) = 3.8x, where x represents the number of kilometers Maddie has run.

19. If Maddie continues at this rate, will she beat her previous best time of 37.5 minutes? Explain your answer.
See margin

14. *f(x) = 704 – 19.9x or f(x) = -19.9x + 704, where x represents the time (in hours) since Bob started tracking the hurricane.*

15. *The slope, -19.9, means that the distance of the hurricane to land is decreasing by 19.9 miles each hour. The y-intercept, 704, gives us the original distance from land when Bob started tracking the hurricane.*

19. *No.*

f(10) = 3.8(10) = 38. She will finish the race in about 38 minutes, which is a half minute slower than her previous best time.

Use the following situation to answer problems 20 – 21.

FINANCE

Nikki has a job as a waitress where she gets an hourly wage plus tips. The table below shows her total earnings for working one weekend.

TIME WORKED (HOURS)	1	2	3	4	5
TOTAL EARNED (DOLLARS)	9.3	19.3	30.73	43.23	56.33

Nikki calculates that the function $f(x) = 11.8x - 2.5$ models her earnings over time. She understands that the slope of 11.8 means she earned an average of about $11.80 per hour. However, she is uncertain about why she has a negative y-intercept in her function equation, since she didn't earn -$2.50 for working 0 hours.

20. Is her equation correct? Explain why or why not.
See margin

21. Since she earned $0 for working zero hours, she now decides to include the point (0, 0) in her data set. How will this affect her function equation to model the situation?
See margin

20. *Yes. Her equation models the data closely. Not every point contained in the function will fit the actual data; it is only an approximation of the relationship between the data*

21. *Her new equation will be approximately $f(x) = 11.3x$. This equation also models the data closely, but now the function goes through the point (0, 0) specifically.*

36 CHAPTER 1: ALGEBRAIC PATTERNS