TEKS

AR.2A Determine the patterns that identify the relationship between a function and its common ratio or related finite differences as appropriate, including linear, quadratic, cubic, and exponential functions.

A.12C Identify terms of arithmetic and geometric sequences when the sequences are given in function form using recursive processes.

A.12D Write a formula for the *n*th term of arithmetic and geometric sequences, given the value of several of their terms.

MATHEMATICAL PROCESS SPOTLIGHT

AR.1D Communicate mathematical ideas, reasoning, and their implications using multiple representations, including symbols, diagrams, graphs, and language as appropriate.

ELPS

3G Express opinions, ideas, and feelings ranging from communicating single words and short phrases to participating in extended discussions on a variety of social and grade-appropriate academic topics.

VOCABULARY

arithmetic sequence, geometric sequence, common difference, common ratio, recursive, additive relationship, multiplicative relationship

1.1 Arithmetic and Geometric Sequences

FOCUSING QUESTION How are arithmetic and geometric sequences alike? How are they different?

LEARNING OUTCOMES

- I can determine patterns that identify a linear function or an exponential function.
- I can identify terms of an arithmetic or geometric sequence. (Algebra 1)
- I can write a formula for the n^{th} term of an arithmetic or geometric sequence. (Algebra 1)
- I can use symbols, tables, and language to communicate mathematical ideas.

ENGAGE

Brenda is at the farmer's market. There are several baskets of tomatoes on a table. Each basket contains 6 tomatoes. What sequence would Brenda create if she listed the number of tomatoes in a set of baskets (1 basket, 2 baskets, 3 baskets, etc.)?

6, 12, 18, 24, 30, …

EXPLORE

The first few terms of two different sequences are shown.

SEQUENCE 1

SEQUENCE 2

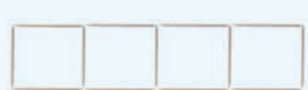
Use toothpicks or counters to build the next two terms of each sequence. Record your information in the table.

See margin.

MATERIALS

- 50 toothpicks for each student group
- 65 two-color counters (or other round objects like pennies) for each student group

SEQUENCE 1:



Term 4

Term 5

SEQUENCE 2:

Term 5

16 counters

Term 6

32 counters

SEQUENCE 1	
TERM NUMBER	NUMBER OF TOOTHPICKS
1	4
2	7
3	10
4	13
5	16
6	19
10	31

SEQUENCE 2	
TERM NUMBER	NUMBER OF COUNTERS
1	1
2	2
3	4
4	8
5	16
6	32
10	512

1. What patterns do you observe in Sequence 1?
   The number of toothpicks increases by 3 (add 3) for the next term.

2. What relationships do you observe between the term number and the number of toothpicks required to build each term in Sequence 1? *(Hint: Record the multiples of 3 next to the number of toothpicks.)*
   See margin

3. How does the pattern that you mentioned in question 2 appear in the figures that you constructed?
   See margin

4. How many toothpicks would you need to build the 10^{th} term of Sequence 1?
   31

5. What patterns do you observe in Sequence 2?
   The number of counters doubles (multiply by 2) for the next term.

6. What relationships do you observe between the term number and the number of counters required to build each term in Sequence 2? *(Hint: Record powers of 2 next to the number of counters.)*
   See margin

7. How does the pattern that you mentioned in question 6 appear in the figures that you constructed?
   See margin

8. How many counters would you need to build the 10^{th} term of Sequence 2?
   512

9. Compare the two sequences. How are they alike? How are they different?
   See margin

2. *Possible answer: The number of toothpicks was 1 more than 3 times the term number.*

3. *Possible answer based on response to previous question:*
For the next term, I started with what I had before and then added 3 more toothpicks to place another square onto the end of the term. The first term begins with 1 toothpick on the left edge of the square and 3 toothpicks added to complete the square.

Note: additional patterns are possible, such as n sets of 2 horizontal toothpicks and n + 1 vertical toothpicks.

6. *The number of counters is one power of 2 less than the power of two that corresponds to the term number.*

7. *For the next term, I counted the number of counters in the previous term and built a stack that was twice as high.*

9. *In Sequence 1, you add 3 to the previous term to get the next term. In Sequence 2, you multiply the previous term by 2 to get the next term. Both sequences have a recognizable pattern, but Sequence 1 is an additive relationship while Sequence 2 is a multiplicative relationship. Sequence 2 grows much faster.*

INSTRUCTIONAL HINTS

Comparing and Contrasting is a high-yield instructional strategy identified by Robert Marzano and his colleagues (*Classroom Instruction That Works*, 2001). To support this strategy, consider asking students to make a Venn diagram comparing properties of Sequence 1 to properties of Sequence 2. This strategy helps students identify differences between linear and exponential relationships, which they will spend time during this chapter studying.

SUPPORTING ENGLISH LANGUAGE LEARNERS

Expressing ideas and opinions (ELPS: 3G) is an important part of student-student discourse about mathematics. Placing English language learners in small groups with their peers, as in the activity in this lesson, provides a safe environment for them to sharpen their skills using the English language while they are learning about mathematics.

REFLECT ANSWERS:

Sequence 1 has a constant difference because to build or get to the next term, you add 3 toothpicks to the previous term every time.

Sequence 2 has a constant ratio because to build or get to the next term, you multiply the previous number of counters by 2 every time.

QUESTIONING STRATEGY

In mathematics, difference is defined as the result of subtraction, yet in the "explain" problem, the common difference is 4. Why is an "addend of 4" considered a difference?

REFLECT

- **Which sequence has a constant difference between the numbers of objects for successive terms? How do you know?**
  See margin.
- **Which sequence has a constant ratio between the numbers of objects for successive terms? How do you know?**
  See margin.

EXPLAIN

A sequence is a set of numbers that are listed in order and that follow a particular pattern. An **arithmetic sequence** is a sequence that has a constant difference between consecutive terms.

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For example, suppose Bernardo earns tickets from the local arcade. He starts out with 11 tickets and earns 4 more tickets for each game he plays. Bernardo can create an arithmetic sequence to show the number of tickets he has after each game.

11, 15, 19, 23 27, 31, ...

+4 +4 +4 +4 +4

Notice that the next term in the sequence can be generated by adding 4 to the previous term. You can use recursive notation to generalize an arithmetic sequence like Bernardo's.

$$a_1 = 11$$
$$a_n = a_{n-1} + 4$$

The addend of 4 is used to generate the next term in this arithmetic sequence. In general, the addend in an arithmetic sequence is called a **common difference**, since it is also the difference between consecutive terms.

A **subscript** is used to indicate a special case of a variable. a_1 indicates the first term of a sequence, a_2 indicates the second term of a sequence, and so on. a_1 is read "*a*-sub one," where "sub" indicates a subscript.

4 CHAPTER 1: ALGEBRAIC PATTERNS

A **geometric sequence** is a sequence that has a constant ratio between consecutive terms. For example, Kayla earns \$0.02 the first week for her allowance, but each week she earns twice as much as she did the week before. She can create a geometric sequence to show the amount of money she earns each week through her allowance.

\$0.02	\$0.04	\$0.08	\$0.16	\$0.32	\$0.64	\$1.28	\$2.56	\$5.12	\$10.24	...
×2	×2	×2	×2	×2	×2	×2	×2	×2		

The multiplier of 2 is used to generate the next term in this geometric sequence. In general, the multiplier in a geometric sequence is called a **common ratio**, since it is also the ratio between consecutive terms.

Kayla's sequence can be represented by a recursive rule.

$$a_1 = 0.02$$
$$a_n = 2a_{n-1}$$

ARITHMETIC AND GEOMETRIC SEQUENCES

An arithmetic sequence has a constant addend or common difference. It is an additive relationship between terms of the sequence.

A geometric sequence has a constant multiplier or common ratio. It is a multiplicative relationship between terms of the sequence.

If you have a sequence, you can use the constant difference or constant multiplier to determine subsequent terms of the sequence.

EXAMPLE 1

Anthony worked all summer to save \$1950 for spending money during the school year. He plans to withdraw the same amount from his savings account at the end of each week. Anthony can create an arithmetic sequence that shows the balance of his savings account at the beginning of each week of the school year.

\$1950, \$1885, \$1820

How much money will be in Anthony's savings account at the beginning of the fourth week of the school year? How much money will be in Anthony's savings account at the beginning of the fifth week of the school year?

INSTRUCTIONAL HINTS

Students process new skills better when they write about them in their own words. Take a moment to have students write about the difference between arithmetic and geometric sequences. Then have students share their explanations with one or two other students.

ADDITIONAL EXAMPLE

Consider the scenario: Ericka and her mother bake 120 cookies for a bake sale. Every 15 minutes they sell half of the cookies on the table. They can create a geometric sequence to show their cookie sale.

120, 60, 30, 15, …

Ask students what the constant multiplier in this sequence would be. Listen for students to say "divide by 2." Ask them how to write "divide by 2" as multiplication.

The multiplier of $\frac{1}{2}$ is used to generate the next term in the sequence.

Ask students to write a recursive rule for the sequence.

$a_1 = 120$

$a_n = \frac{1}{2}a_{n-1}$

Ask students to continue the sequence. Discuss reasonableness of terms 5 and beyond. Would a customer buy half of a cookie?

$a_5 = 7.5,\ a_6 = 3.75$

ADDITIONAL EXAMPLES

1. Daniel's grandmother gave him a $50 gift card for music downloads for his birthday. If he downloads one song per day at the same price, he can create an arithmetic sequence that shows the balance of his gift card each day.

$50, $48.55, $47.10, $45.65,...

What is the common difference, and what does it represent? How much money will be left on Daniel's gift card at the end of one week?

The common difference is -$1.45.

At the end of one week (7 days), Daniel will have $39.85 on his gift card.

2. Keniesha went for a hike. She hiked at a steady pace, noting her distance, in miles, every 15 minutes. She can create an arithmetic sequence to reflect her distance traveled every 15 minutes of her hike.

0.75, 1.5, 2.25, 3, …

What is the common difference, and what does it represent? How far will Keniesha have hiked if she hikes for one and a half hours?

The common difference is 0.75. It represents the distance in miles that Keniesha hiked every 15 minutes. In one and a half hours, Keniesha will have hiked 4.5 miles.

STEP 1 First, use the existing data to determine the common difference.

TIME (WEEKS)	SAVINGS BALANCE (DOLLARS)
1	$1950
2	$1885
3	$1820

–$65 = $1885 – $1950

–$65 = $1820 – $1855

The common difference is –$65.

STEP 2 Next, apply the common difference to the balance at the beginning of the third week to determine the balance at the beginning of the fourth week.

$$\$1820 + (-\$65) = \$1755$$

STEP 3 Then, apply the common difference to determine the balance in Anthony's savings account at the beginning of the fifth week of school.

$$\$1755 + (-\$65) = \$1690$$

The balance in Anthony's savings account is $1755 at the beginning of the fourth week of school and $1690 at the beginning of the fifth week of school.

YOU TRY IT! #1

Rachel helps the student council create a paper chain that contains students' written pledges not to bully or tolerate bullying. When Rachel begins stapling, there are 54 inches of paper chain. She measures after adding each link and records the results in a table.

NUMBER OF LINKS RACHEL ADDED	1	2	3
LENGTH OF PAPER CHAIN (INCHES)	$58\frac{1}{2}$	63	$67\frac{1}{2}$

What is the common difference in this situation, and how long will the paper chain be after Rachel has added a total of six links to it?

See margin.

YOU TRY IT! #1 ANSWER:

The common difference is $4\frac{1}{2}$ and the paper chain will be 81 inches long after Rachel has added six links to it.

QUESTIONING STRATEGY FOR YOU TRY IT #1

Some students may expect the table of values to start with (1, 54). Ask students why the dependent variable starts with $58\frac{1}{2}$ rather than 54. What column could be added to the table to account for the 54 inches of paper chain that Rachel started with?

EXAMPLE 2

At the beginning of the year, an investor puts $1000 into a fund that pays 20% annually. The investor projects how much will be in the fund at the end of each year for the next three years.

$1200, $1440, $1728

How much money will be in the investment fund at the end of four years? How much money will be in the investment fund at the end of five years?

STEP 1 First, determine the common ratio in this situation.

$1440 ÷ $1200 = 1.2
$1728 ÷ $1440 = 1.2
The common ratio is 1.2.

STEP 2 Next, multiply the third value in the geometric sequence by the common ratio to determine the fourth value in the geometric sequence.

($1728)(1.2) = $2073.60

STEP 3 Then, multiply by the common ratio to determine the fifth value in the geometric sequence.

($2073.60)(1.2) = $2488.32

There will be $2073.60 in the fund after four years and $2488.32 in the fund after five years.

QUESTIONING STRATEGIES

How is 20% written as a decimal?

The fund pays 20% annually, yet the common ratio is 1.2, not 0.2. What does 1.2 represent?

ADDITIONAL EXAMPLES

1. Jack purchased a new car for $25,120. Each year the value of the car will decrease by 30%. Jack projected how much his car will be worth at the end of each year for the next four years.

$25,120, $17,584, $12,308.80, $8,616.16

What is the common ratio in this situation? Is the common ratio what you expected? Why or why not?

The common ratio is 0.7. It represents 70% of the value of Jack's car from the previous year.

What will the value of Jack's car be after 8 years?

If Jack's car continues to decrease in value by 30% each year, it will be worth $1,448.12 at the end of 8 years.

2. Montrell gets a job icing cupcakes for a local bakery. On his first day on the job, he is proud to notice that the more cupcakes he ices, the faster he gets. Each hour, he takes note of how many cupcakes he has iced in that hour.

5, 10, 20, …

If Montrell continues icing cupcakes at his current rate, how many cupcakes will he ice in his fifth hour at the bakery? How many total cupcakes will he have iced by the end of his 6-hour shift?

Montrell will ice 80 cupcakes in his 5th hour. At the end of his 6-hour shift, he will have iced 315 cupcakes.

YOU TRY IT! #2 ANSWER:

The common ratio is 0.8 and there are approximately 164 milligrams of medicine in the patient's system after five hours.

QUESTIONING STRATEGIES

Assist students' thinking as they process the difference between recursive and explicit rules by asking the following questions.

- Explain how a recursive rule differs from an explicit rule?
- For which type of rule must you know the previous term in order to find the next?
- For which type of rule can you find any term in the sequence without needing to continue the sequence?
- Which rule do you prefer and why?
- How does the explicit rule relate to a function rule?

YOU TRY IT! #2

A patient takes 500 milligrams of medicine. A nurse charts the amount of medication in the patient's system.

TIME SINCE DOSAGE (HOURS)	MEDICINE IN PATIENT'S SYSTEM (MG)
1	400
2	320
3	256
4	204.8

What is the common ratio in this situation and approximately how much medicine, to the nearest milligram, will remain in the patient's system after five hours?

See margin.

EXAMPLE 3

For the sequence shown, write a recursive rule and an explicit rule.

4, 6.5, 9, 11.5, 14, …

A **recursive rule** shows how to determine the nth term, a_n, using the value of the previous term. An **explicit rule**, like a function, shows how to determine the nth term, a_n, using the term number, n.

STEP 1 **First, determine the common difference or ratio in this situation.**

4, 6.5, 9, 11.5, 14, …

+2.5 +2.5 +2.5 +2.5

The common difference is +2.5.

STEP 2 **Next, write the first term in the sequence, a_1. Use the common difference to write a recursive rule relating a_n to the previous term, a_{n-1}.**

$$a_1 = 4$$
$$a_n = a_{n-1} + 2.5$$

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STEP 3 Use the common difference to work backwards to determine the value of term 0.

1.5, 4, 6.5, 11.5, 14, ...

+2.5

STEP 4 Use the common difference and the value of term 0 to write an explicit rule with term 0 as the starting point and the common difference as the rate of change.

$$a_n = 1.5 + 2.5n$$

YOU TRY IT! #3

Write a recursive rule and an explicit rule for the sequence 6, $7\frac{1}{3}$, $8\frac{2}{3}$, 10, $11\frac{1}{3}$, …

See margin.

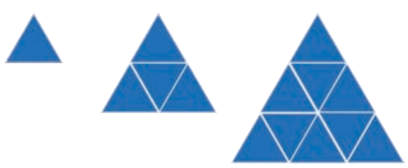
PRACTICE/HOMEWORK

For questions 1 – 4 write an explicit rule that describes the number of items used to construct the pattern in terms of the term number, n.

1.

$a_n = 2 + 2n$

2.



$a_n = n^2$

ADDITIONAL EXAMPLES

Write the recursive, explicit, and function rules for the additional examples on page. 7.

1. Jack's car

recursive rule:
$a_1 = 25{,}120,\ a_n = 0.70a_{n-1}$

explicit rule:
$a_n = 25{,}120(0.70)^n$

function rule:
$f(x) = 25{,}120(0.70)^x$

2. Montrell's job

recursive rule:
$a_1 = 5,\ a_n = 2a_{n-1}$

explicit rule:
$a_n = 2.5(2)^n$

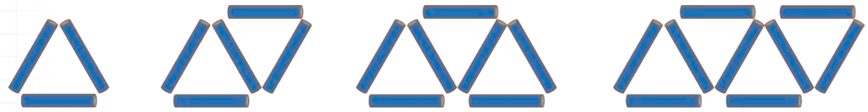
function rule:
$f(x)=2.5(2)^x$

YOU TRY IT! #3 ANSWER:

recursive rule:
$a_1 = 6,\ a_n = a_{n-1} + 1\frac{1}{3}$

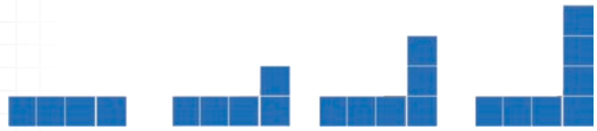
explicit rule:
$a_n = 4\frac{2}{3} + 1\frac{1}{3}n$

3.



$a_n = 1 + 2n$

4.



$a_n = 3 + n$

For questions 5 and 6, use the following situation.

FINANCE

Segway Tours in Corpus Christi charges $12 an hour to rent a Segway and an additional fee of $4 for the required helmet. David can create an arithmetic sequence that shows the cost of renting a Segway.

16, 28, 40, 52, …

5. How much will David spend to rent the Segway with a helmet for 6 hours?
$76.00

6. Write a function rule that describes the cost of renting a Segway, $f(n)$, in terms of the number of hours, n, David rents the Segway.
$f(n) = 4 + 12n$

For questions 7 and 8, use the following situation.

SCIENCE

Roger dropped a ball from a height of 1000 centimeters. The height of the ball is 80% of the previous height after each bounce of the ball. Roger can create a geometric sequence that shows the height of the ball at the end of each bounce.

800, 640, 512, 409.6, …

7. What is the height of the ball after the 5^{th} bounce?
327.68 centimeters

8. Write a function rule that describes the height of the ball, $f(n)$, after the number of bounces, n, the ball makes.
$f(n) = 1000(0.8)^n$

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For questions 9 and 10, use the following situation.

FINANCE

Clayton opens a savings account with $11 he got from his grandmother. Each month after the initial deposit, he adds $15 to the account. Clayton can create an arithmetic sequence that shows the balance of his savings account at the end of each month after he deposits funds in the savings account.

26, 41, 56, …

9. How much money will Clayton have in his account after he deposits money for 12 months?
$191.00

10. Write an explicit rule that describes the amount of money in Clayton's account, a_n, in terms of the number of months, n, he deposits money.
$a_n = 11 + 15n$

For questions 11 – 16 determine whether the sequences shown are arithmetic or geometric sequences. Then, write a recursive rule and an explicit rule.

11. 1, 8, 15, 22, 29, …
See margin.

12. 2, 6, 18, 54, 162, …
See margin.

13. −10, −6.5, −3, 0.5, 4, 7.5, …
See margin.

14. 1.5, 7.5, 37.5, 187.5 …
See margin.

15. 64, 16, 4, 1, 0.25, …
See margin.

16. 147, 127, 107, 87, 67, …
See margin.

For questions 17 – 20 for each recursive rule and explicit rule given below, write the first 4 terms in the sequence.

17. $a_1 = 9.5;\ a_n = a_{n-1} + 6.5$
$a_n = 3 + 6.5n$
9.5, 16. 22.5, 29

18. $a_1 = 3;\ a_n = 4a_{n-1}$
$a_n = 3(4)^{n-1}$
3, 12, 48, 192

19. $a_1 = 625;\ a_n = a_{n-1} \div 5 = \frac{1}{5}a_{n-1}$
$a_n = 625\left(\frac{1}{5}\right)^{n-1}$
625, 125, 25, 5

20. $a_1 = 140;\ a_n = a_{n-1} - 30$
$a_n = 170 - 30n$
140, 110, 80, 50

11. *arithmetic*
$a_1 = 1;\ a_n = a_{n-1} + 7$
$a_n = -6 + 7n$

12. *geometric*
$a_1 = 2;\ a_n = 3a_{n-1}$
$a_n = 2(3)^{n-1}$

13. *arithmetic*
$a_1 = -10;\ a_n = a_{n-1} + 3.5$
$a_n = -13.5 + 3.5n$

14. *geometric*
$a_1 = 1.5;\ a_n = 5a_{n-1}$
$a_n = 1.5(5)^{n-1}$

15. *geometric*
$a_1 = 64;\ a_n = a_{n-1} \div 4 = \frac{1}{4}a_{n-1}$
$a_n = 64\left(\frac{1}{4}\right)^{n-1}$

16. *arithmetic*
$a_1 = 147;\ a_n = a_{n-1} - 20$
$a_n = 167 - 20n$