6.1 Representing Data in Matrices

FOCUSING QUESTION What is a matrix, and how do I use it to represent and organize a set of data?

LEARNING OUTCOMES

- I can use a matrix to represent and organize a data set.
- I can apply mathematics to solve problems arising in everyday life, society, and the workplace.

ENGAGE


pixabay.com

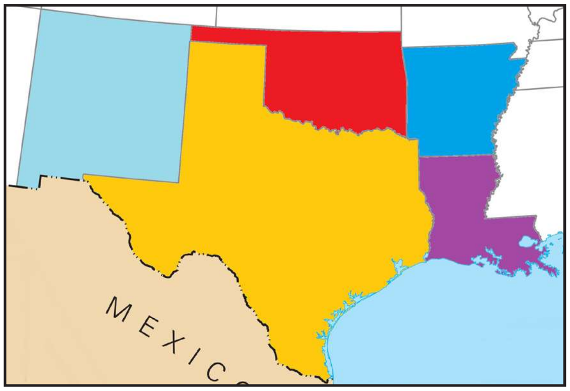
A trilogy is a set of three. Popular movie trilogies have generated hundreds of millions of dollars in box office sales. The list below shows the worldwide box office revenue for certain movie trilogies.

- The Matrix (1999), $463,420,706; The Matrix Reloaded (Part 2, 2003), $738,576,929; The Matrix Revolutions (Part 3, 2003), $427,289,109
- Star Wars, Episode IV (1977), $786,535,665; Star Wars, Episode V (1980), $534,058,751; Star Wars, Episode VI (1983), $572,625,409
- Jurassic Park (1993), $1,038,812,584; The Lost World: Jurassic Park (Part 2, 1997), $618,626,844; Jurassic Park III (2001), $365,900,000

Create an array where each row represents a trilogy and each column represents the worldwide box office revenue for Part 1, Part 2, and Part 3 of that movie.

EXPLORE

The list below contains the populations in 2014, according to the U.S. Census Bureau, of the five largest cities in Texas and each adjacent state.



- Arkansas: Little Rock (197,706), Fort Smith (87,351), Fayetteville (80,621), Springdale (76,565), Jonesboro (72,210)
- Louisiana: New Orleans (384,320), Baton Rouge (228,895), Shreveport (198,242), Lafayette (126,066), Lake Charles (74,889)

- New Mexico: Albuquerque (557,169), Las Cruces (101,408), Rio Rancho (93,820), Santa Fe (70,297), Roswell (48,608)
- Oklahoma: Oklahoma City (620,602), Tulsa (399,682), Norman (118,040), Broken Arrow (104,726), Lawton (97,017)
- Texas: Houston (2,239,558), San Antonio (1,436,697), Dallas (1,281,047), Austin (912,791), Fort Worth (812,238)

1. Work with a partner. Write the population of each city on a separate sticky note or piece of paper. Arrange the sticky notes in an array so that each row represents one state and each column represents the population of the largest, 2nd largest, 3rd largest, 4th largest, and 5th largest cities in that state. To better organize your arrangement, you may wish to place the rows so that the states are listed alphabetically.

2. Record the array on a piece of paper. Place large brackets around the left and right sides of your array.

A **matrix** is a rectangular array with data organized into rows and columns. Brackets are used to indicate the boundaries of a matrix. Entries in a matrix are identified by the row and column in which they appear.

3. In your matrix, identify the entry in Row 2, Column 3 (notated: $a_{2,3}$). In this notation, a represents an entry in matrix A.

4. Use matrix notation to identify the entry that represents the population of Austin, Texas, and the entry that represents Shreveport, Louisiana.

5. The list below shows the number of regular season games won by 6 university football teams for 4 recent years. Create a matrix for the data set shown by placing the universities in each row and the years in each column.
- University of Texas: 9 (2012), 8 (2013), 6 (2014), 5 (2015)
- Texas A&M University: 11 (2012), 9 (2013), 8 (2014), 8 (2015)
- University of Michigan: 8 (2012), 7 (2013), 5 (2014), 9 (2015)
- University of Arkansas: 4 (2012), 3 (2013), 7 (2014), 7 (2015)
- University of Louisville: 11 (2012), 12 (2013), 9 (2014), 7 (2015)
- Stanford University: 12 (2012), 11 (2013), 8 (2014), 11 (2015)

6. In which row and column is the entry 3?

7. What is the entry for $a_{6,3}$?

8. How does the shape of the football matrix compare to the shape of the largest cities matrix?

REFLECT

- **How does a matrix help you to organize data?**
- **How can you represent a data set in a matrix?**

EXPLAIN

A matrix is a way to record, organize, and represent data. Like an array, a matrix uses rows and columns to organize the data. Each matrix entry corresponds with one row and one column and can be identified by its row and column location, much like a coordinate system.

> The word **matrix** is singular, meaning only one matrix. The word **matrices** is plural and used when describing more than one matrix.

For example, the table shows the number of cheeseburgers and chicken nugget baskets sold in the school cafeteria for each of five days during one week.

The same data set could also be represented in a matrix. Each row represents a day of the week. Each column represents one food item: the first column represents the number of cheeseburgers sold and the second column represents the number of chicken nugget baskets sold.

DAY	NUMBER OF CHEESE-BURGERS SOLD	NUMBER OF CHICKEN NUGGET BASKETS SOLD
MONDAY	97	86
TUESDAY	103	105
WEDNESDAY	110	97
THURSDAY	85	113
FRIDAY	120	98

$$\begin{array}{cc} \text{Cheeseburgers} & \text{Chicken Nuggets} \\ \end{array}$$

$$\begin{bmatrix} 97 & 86 \\ 103 & 105 \\ 110 & 97 \\ 85 & 113 \\ 120 & 98 \end{bmatrix} \quad \text{Day of the Week}$$

A matrix can have a name, such as Matrix A. The dimensions of a matrix are given by the number of rows and number of columns, respectively. If matrix A is the school cafeteria matrix, then matrix A is a 5×2 matrix since it has **5 rows** and **2 columns**.

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The rows and columns in a matrix are used to identify the data entries. . The row and column are used like coordinates to identify a particular entry.

$$\begin{bmatrix} 97 & 86 \\ 103 & 105 \\ 110 & 97 \\ 85 & 113 \\ 120 & 98 \end{bmatrix} \longleftrightarrow \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \\ a_{4,1} & a_{4,2} \\ a_{5,1} & a_{5,2} \end{bmatrix}$$

The letter a is used to represent an entry in matrix A. Subscripts are used to indicate the row and column, in order, of the location of the entry. The entry in matrix A for Tuesday's number of cheeseburgers sold, 103, is $a_{2,1}$ since it is located in **Row 2** and **Column 1**.

Matrices are especially useful for performing calculations on large data sets. Computer programs and numerical models use matrices extensively because the procedures for computing with matrices are very routine. In the rest of this chapter, you will investigate how to add, subtract, and multiply data in matrices.

REPRESENTING DATA IN A MATRIX

A matrix is a rectangular array of numbers that are arranged in rows and columns.

- The dimensions of a matrix are the number of rows by the number of columns. A matrix with 6 rows and 4 columns is a 6×4 matrix.
- Each row is one category and each column is a category.
- Data elements, or *entries*, in one row must all share the same category.
- Entries in one column must all share the same category.
- Entries are identified by their row and column numbers. The entry of matrix A that is found in row 3 and column 1 is identified by $a_{3,1}$.

EXAMPLE 1

Organize the data about the box office earnings of the Star Wars movies into a matrix. The earnings are rounded to nearest tenth of a million dollars. Explain your categories and organizational layout. Identify the dimensions of the matrix you have constructed.

RELEASE DATE	MOVIE	DOMESTIC OPENING WEEKEND	DOMESTIC BOX OFFICE	WORLDWIDE BOX OFFICE
MAY 25, 1977	STAR WARS EP. IV: A NEW HOPE	1.5	460.9	786.5
MAY 21, 1980	STAR WARS EP. V: THE EMPIRE STRIKES BACK	4.9	290.2	534.1
MAY 25, 1983	STAR WARS EP. VI: RETURN OF THE JEDI	23.0	309.1	572.6
MAY 19, 1999	STAR WARS EP. I: THE PHANTOM MENACE	64.8	474.5	1,027.0
MAY 16, 2002	STAR WARS EP. II: ATTACK OF THE CLONES	80.0	302.2	648.2
MAY 19, 2005	STAR WARS EP. III: REVENGE OF THE SITH	108.4	380.3	849.0
DEC 18, 2015	STAR WARS EP. VII: THE FORCE AWAKENS	248.0	740.3	1,510.8

Source: Nash Information Services, LLC

STEP 1 **Decide what category you could use to organize the data in rows.**

You could place the movies as categories in the rows, as they are in order of their release date in the table or arranged numerically by episode.

STEP 2 **Decide what category you could use to organize the data in columns.**

You could place the three categories of earnings: domestic opening weekend, domestic box office, and worldwide box office, across the columns.

STEP 3 **Enter the amounts in the matrix according to your layout decisions. Units, including dollar signs, are not included in matrices. Add brackets. The titles of the rows and columns are not part of the matrix. Shown is one possible matrix.**

$$\begin{array}{rc}
 & \begin{array}{ccc} \text{Domestic Opening Box Office} & \text{Domestic Box Office} & \text{Worldwide Box Office} \end{array} \\
\begin{array}{r} \text{Star Wars IV} \\ \text{Star Wars V} \\ \text{Star Wars VI} \\ \text{Star Wars I} \\ \text{Star Wars II} \\ \text{Star Wars III} \\ \text{Star Wars VII} \end{array} &
\begin{bmatrix} 1.5 & 460.9 & 786.5 \\ 4.9 & 290.2 & 534.1 \\ 23.0 & 309.1 & 572.6 \\ 64.8 & 474.5 & 1{,}027.0 \\ 80.0 & 302.2 & 648.2 \\ 108.4 & 380.3 & 849.0 \\ 248.0 & 740.3 & 1{,}510.8 \end{bmatrix}
\end{array}$$

STEP 4 **Determine the dimensions of your matrix.**

If your columns are the categories of earnings and your rows are categorized by the movies, the matrix would be 7 rows by 3 columns or 7×3 as in the example in Step 2. Instead, if your columns are categorized by the movies and the rows are the earnings categories, the matrix would be 3 rows by 7 columns or 3×7.

YOU TRY IT! #1

Choose some of the data from the chart and organize the data you've chosen into a matrix. Explain your choices and how you organized the categories, including any rounding. Identify the dimensions of the matrix you have constructed.

BOX OFFICE COMPARISON FOR ALL-TIME TOP-GROSSING FILMS

MOVIE	PRODUCTION BUDGET	DOMESTIC OPENING WEEKEND	DOMESTIC BOX OFFICE	WORLDWIDE BOX OFFICE
TITANIC	$200,000,000	$28,638,131	$658,672,302	$2,207,615,668
AVATAR	$425,000,000	$77,025,481	$760,507,625	$2,783,918,982
THE AVENGERS	$225,000,000	$207,438,708	$623,279,547	$1,519,479,547
JURASSIC WORLD	$215,000,000	$208,806,270	$652,198,010	$1,670,328,025
STAR WARS EP. VII: THE FORCE AWAKENS	$200,000,000	$247,966,675	$740,265,583	$1,510,765,583

Source: Nash Information Services, LLC

EXAMPLE 2

Given Matrix A_1 showing the American Football Conference South Division standings as of week 17 in the 2015 season, identify what the data entries in $a_{4,2}$, $a_{1,3}$, and $a_{3,1}$ represent.

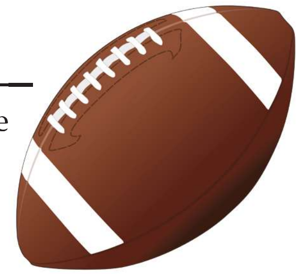


Image source: openclipart.org

$$\begin{array}{cc} & \begin{array}{ccc} \text{Wins} & \text{Losses/Ties} & \text{Percent Won} \end{array} \\ A_1 = & \begin{bmatrix} 9 & 7 & 56.3 \\ 8 & 8 & 50.0 \\ 5 & 11 & 31.3 \\ 3 & 13 & 18.8 \end{bmatrix} \begin{array}{l} \text{Houston Texans} \\ \text{Indianapolis Colts} \\ \text{Jacksonville Jaguars} \\ \text{Tennessee Titans} \end{array} \end{array}$$

Source: NFL.com

STEP 1 **Recall that $a_{4,2}$ means the entry in the 4th row and 2nd column.**

The data in that location are the losses and ties for the Tennessee Titans (13 losses/ties).

STEP 2 **Determine the entry in $a_{1,3}$.**

The data in the 1st row and 3rd column shows the percent of games won by the Houston Texans (56.3%: 9 wins compared to 16 games in all).

STEP 3 **Determine the entry in $a_{3,1}$.**

This is not the same as the entry in $a_{1,3}$. The data in the 3rd row and 1st column are the wins for the Jacksonville Jaguars (5 wins).

YOU TRY IT! #2

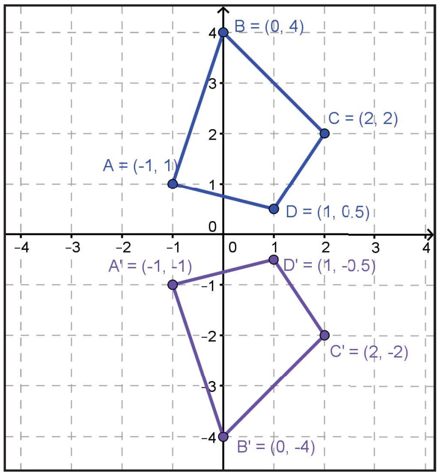
Given Matrix A_2 showing the American Football Conference East Team standings as of week 17 in the 2015 season, identify what the data entries in $a_{2,3}$, $a_{4,1}$, and $a_{3,2}$ represent.

$$\begin{array}{cc} & \begin{array}{ccc} \text{Wins} & \text{Losses/Ties} & \text{Percent Won} \end{array} \\ A_2 = & \begin{bmatrix} 12 & 4 & 75.0 \\ 10 & 6 & 62.5 \\ 8 & 8 & 50.0 \\ 6 & 10 & 37.5 \end{bmatrix} \begin{array}{l} \text{New England Patriots} \\ \text{New York Jets} \\ \text{Buffalo Bills} \\ \text{Miami Dolphins} \end{array} \end{array}$$

Source: NFL.com

EXAMPLE 3

One use of matrices is to describe transformations of geometric figures. On the graph, quadrilateral *ABCD* has been reflected over the *x*-axis. The coordinates of each figure are listed. Create two matrices, one for each list, and compare the data sets.



$A = (-1, 1)$ $\quad A' = (-1, -1)$
$B = (0, 4)$ $\quad B' = (0, -4)$
$C = (2, 2)$ $\quad C' = (2, -2)$
$D = (1, 0.5)$ $\quad D' = (1, -0.5)$

STEP 1 Make a matrix, M_1, for quadrilateral *ABCD*.

$$M_1 = \begin{array}{c} \\ x \\ y \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} -1 & 0 & 2 & 1 \\ 1 & 4 & 2 & 0.5 \end{bmatrix} \end{array}$$

STEP 2 Make a matrix, M_2, for quadrilateral $A'B'C'D'$.

$$M_2 = \begin{array}{c} \\ x \\ y \end{array} \begin{array}{c} \begin{array}{cccc} A' & B' & C' & D' \end{array} \\ \begin{bmatrix} -1 & 0 & 2 & 1 \\ -1 & -4 & -2 & -0.5 \end{bmatrix} \end{array}$$

STEP 3 Describe the layout of the matrices and compare the data in them.

The first row contains the *x*-coordinates and the second row contains the *y*-coordinates of each of the vertices. The columns show the names of the vertices for each figure. Both are 2×4 matrices. The data sets are the same except the *y*-coordinates are opposites.

YOU TRY IT! #3

Use the tables for the sales after Thanksgiving Day at the Que Cute Boutique to construct a matrix for each. Make a 3×2 matrix for the sales for weeks 1 through 3 and a 2×3 matrix for the sales for weeks 4 through 6. Use the data to compare shirt and pants sales and the trend overall for sales.

WEEK	SHIRTS	PANTS
1	$725	$695
2	$540	$485
3	$565	$505

WEEK	SHIRTS	PANTS
4	$805	$725
5	$875	$810
6	$900	$995

PRACTICE/HOMEWORK

Use the matrix shown to answer questions 1 – 4.

Matrix $C = \begin{bmatrix} 88.5 & 74 & 105 & 92.5 \\ 110 & 91.3 & 89 & 133 \\ 181.6 & 112.1 & 76 & 142.6 \end{bmatrix}$

1. What are the dimensions of Matrix C?

2. What is the value of entry $c_{1,2}$?

3. What is the value of entry $c_{2,1}$?

4. What is the value of entry $c_{3,2}$?

Use the data below to answer questions 5 – 7.

HEALTH

PERCENT OF THE POPULATION OVER THE AGE OF 60

COUNTRY	2010	2011	2012
AUSTRALIA	18.9	19.2	19.5
BELIZE	5.6	5.7	5.7
CZECH REPUBLIC	22.4	22.8	23.2
KENYA	4.1	4.2	4.3

Source: World Health Organization (www.who.int)

5. Organize the population data into a matrix.

6. What are the dimensions of your matrix?

7. What percent of the population of Belize was over the age of 60 in the year 2011?

Matrix A shows information about six teams in the Western Conference of the NBA during the month of November, 2015. Use the matrix data to answer questions 8 – 10.

SPORTS

Matrix A:

Wins	Losses/Ties	Percent Won	
9	7	56.3	Dallas Mavericks
2	12	14.3	LA Lakers
9	7	56.3	Oklahoma City Thunder
7	9	43.8	Houston Rockets
13	3	81.3	San Antonio Spurs
6	7	46.2	Utah Jazz

Source: NBA.com

8. Describe the data entry in $a_{5,1}$.

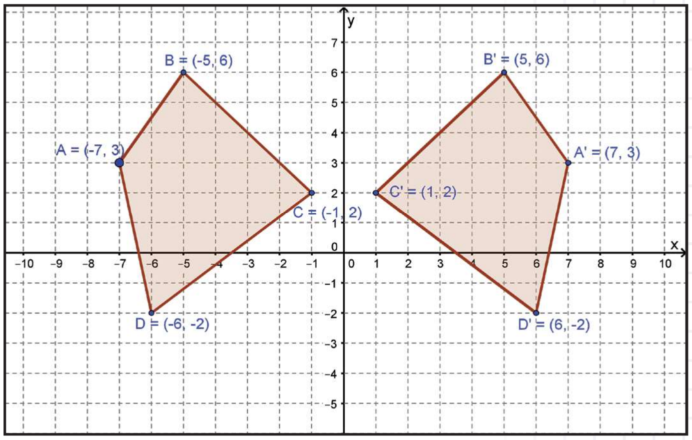
9. Describe the data entry in $a_{1,3}$.

10. Describe the data entry in $a_{3,2}$.

On the graph, quadrilateral ABCD has been reflected over the y-axis. The coordinates of each figure are listed. Use the information to answer questions 11 – 13.

GEOMETRY

$A = (-7, 3)$ $\quad A' = (7, 3)$
$B = (-5, 6)$ $\quad B' = (5, 6)$
$C = (-1, 2)$ $\quad C' = (1, 2)$
$D = (-6, -2)$ $\quad D' = (6, -2)$



11. Create a matrix, M_1, for quadrilateral $ABCD$.

12. Create a matrix, M_2, for quadrilateral $A'B'C'D'$.

13. Compare the data from matrices M_1 and M_2.

The table below shows the monthly sales of different drinks at a concession stand. Use this data to answer questions 14 and 15.

BUSINESS

DRINK	SMALL	MEDIUM	LARGE	EXTRA LARGE
LEMONADE	321	459	324	156
TEA	244	324	143	20
SPORTS DRINK	154	215	63	89
WATER	213	234	368	342
COLA	352	367	547	108

14. Create a 5×4 matrix, M_1, to represent the concession stand data.

15. Create a 4×5 matrix, M_2, to represent the same concession stand data.

Matrix A shows information collected from a customer survey. A customer rated different brands of socks on a scale of 1 to 5 (with 1 being the lowest score). Use this matrix to answer questions 16 – 18.

BUSINESS

Matrix *A*:

$$\begin{array}{c} \\ \text{Brand A} \\ \text{Brand B} \\ \text{Brand C} \\ \text{Brand D} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Comfort} & \text{Cost} & \text{Appearance} \end{array} \\ \begin{bmatrix} 5 & 3 & 4 \\ 4 & 4 & 3 \\ 4 & 2 & 1 \\ 4 & 5 & 5 \end{bmatrix} \end{array}$$

16. What are the dimensions of the matrix?

17. Describe the data entry in $a_{3,1}$.

18. Which brand received the best overall ratings?

Matrix B shows information about the maximum outdoor temperature on certain dates in four Texas cities. Use this matrix to answer questions 19 – 20.

SCIENCE

$$\text{Matrix } B: \quad \begin{array}{l} \\ \text{Houston} \\ \text{Dallas} \\ \text{El Paso} \\ \text{Lubbock} \end{array} \begin{array}{c} \begin{array}{ccc} \text{January 1, 2015} & \text{July 4, 2015} & \text{October 1, 2015} \end{array} \\ \begin{bmatrix} 47^\circ\text{ F} & 92^\circ\text{ F} & 76^\circ\text{ F} \\ 37^\circ\text{ F} & 95^\circ & 66^\circ\text{ F} \\ 46^\circ\text{ F} & 96^\circ\text{ F} & 68^\circ\text{ F} \\ 26^\circ\text{ F} & 91^\circ\text{ F} & 70^\circ\text{ F} \end{bmatrix} \end{array}$$

Source: Weather Underground (wunderground.com)

19. Describe the data entry in $b_{2,1}$.

20. Which city seems to generally be cooler than the others?

6.2 Adding and Subtracting Matrices

FOCUSING QUESTION How do I add and subtract matrices?

LEARNING OUTCOMES

- I can add and subtract data sets that are represented in matrices.
- I can apply mathematics to solve problems arising in everyday life, society, and the workplace.

ENGAGE

An art museum charges non-members a general admission entry fee.

- Adults: $15, Senior/Military: $10, Youth/Student: $7.50

Members receive discounted general admission entry fee.

- Adults: $7.50, Senior/Military: $2.50, Youth/Student: free

How would you determine the amount of money that members save on admission?

EXPLORE

Statistics for the volleyball teams in the Big 12 athletic conference for two recent years are shown in the tables below.

	2014			2015		
UNIVERSITY	ASSISTS	KILLS	SERVICE ACES	ASSISTS	KILLS	SERVICE ACES
BAYLOR	1517	1625	113	1303	1413	130
IOWA STATE	1404	1482	123	1389	1491	159
KANSAS	1513	1627	131	1601	1706	149
KANSAS STATE	1469	1568	126	1392	1510	110
OKLAHOMA	1453	1598	157	1324	1443	108
TEXAS	1324	1426	135	1525	1636	136
TEXAS CHRISTIAN	1450	1542	155	1214	1316	114
TEXAS TECH	1317	1392	110	1227	1308	89
WEST VIRGINIA	1327	1427	97	1091	1178	82

Source: Big 12 Sports

1. Create matrix A to display the statistics from 2014. Create matrix B to display the statistics from 2015. Use each university as a row and each statistic as a column.

2. For Baylor, use number sense or estimation strategies to estimate the combined number of assists, kills, and service aces for both seasons.

3. Create a new matrix S with 9 rows and 3 columns. For the first row, use addition to combine the entries from matrix A and matrix B for Baylor's assists, kills, and service aces. Place the sums in the first row of matrix S in the appropriate columns. Use either paper and pencil or technology to perform the computations.

4. Repeat this process for the remaining universities to complete matrix S. Use either paper and pencil or technology to perform the computations.

5. Based on the context of the data, what do the entries in matrix S represent?

6. Use the matrices to identify the values of $a_{6,3}$ and $b_{6,3}$.

7. Determine $s_{6,3} = a_{6,3} + b_{6,3}$.

8. How are the entries in matrix S, $s_{R,C}$, related to their corresponding entries in matrix A, $a_{R,C}$, and matrix B, $b_{R,C}$?

9. Use graphing technology such as a graphing calculator or app to enter the data from matrix A and matrix B. Remember that matrices are defined by the number of rows by the number of columns.

10. Use matrix operations to add matrix A to matrix B. On some devices, this will appear on the home screen as [A] + [B], and the sum of the two matrices will appear as a new matrix. How does this matrix compare to matrix S, the one you calculated in a previous question?

11. Create a new matrix D with 9 rows and 3 columns. For each entry, use subtraction to determine the difference between the entries from matrix A and matrix B for each university's team's assists, kills, and service aces. Subtract matrix A, which represents 2014 totals, from matrix B, which represents 2015 totals. Place the differences in corresponding entry of matrix D. Use either paper and pencil or technology to perform the computations.

12. Based on the context of the data, what do the entries in matrix D represent?

13. Use the matrices to identify the values of $a_{7,2}$ and $b_{7,2}$.

14. Determine $d_{7,2} = b_{7,2} - a_{7,2}$.

15. How are the entries in matrix D, $d_{R,C}$, related to their corresponding entries in matrix A, $a_{R,C}$, and matrix B, $b_{R,C}$?

16. With graphing technology, use matrix operations to subtract matrix A from matrix B. On some devices, this will appear on the home screen as [B] – [A], and the difference between the two matrices will appear as a new matrix. How does this matrix compare to matrix D, the one you calculated in a previous question?

17. Use technology to compare the sum of matrix A and matrix B, [A] + [B], with the sum of matrix B and matrix A, [B] + [A]. Is addition of matrices commutative (i.e., does the order of the addends matter)?

18. Use technology to compare the difference between matrix A and matrix B, [A] – [B], with the difference between matrix B and matrix A, [B] – [A]. Is subtraction of matrices commutative (i.e., does the order of the two numbers being subtracted matter)?

REFLECT

- **When you add two matrices together, what do you do with the entries in the addend matrices in order to generate the entries for the sum matrix?**
- **When you subtract two matrices, what do you do with the entries in the subtrahend and minuend matrices in order to generate the entries for the difference matrix?**
- **How does adding or subtracting matrices with paper and pencil compare with adding or subtracting matrices using technology?**

EXPLAIN

Matrices are used to organize and represent sets of data. One benefit to using matrices is that once the data is presented in a matrix, you can use technology to efficiently make calculations with the data.

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ADDING MATRICES

To add two matrices together, you must first make sure that the matrices are the same size. The two addend matrices must have the same number of rows and the same number of columns.

> In a matrix entry, the **subscripts** indicate which row and column in which the entry belongs. For example, $a_{3,2}$ indicates an entry from matrix A in row 3, column 2.

In general, to add matrix A and matrix B, you add each pair of corresponding elements.

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \\ b_{3,1} & b_{3,2} \end{bmatrix} = \begin{bmatrix} a_{1,1}+b_{1,1} & a_{1,2}+b_{1,2} \\ a_{2,1}+b_{2,1} & a_{2,2}+b_{2,2} \\ a_{3,1}+b_{3,1} & a_{3,2}+b_{3,2} \end{bmatrix}$$

For example, the table shows the number of each color of mobile phone sold in a store for each of two models of mobile phone. The same data set could also be represented in two matrices. Each row represents a month and each column represents a color.

	MODEL 5			MODEL 5c		
	PINK	GOLD	WHITE	PINK	GOLD	WHITE
LAST MONTH	12	58	37	23	83	62
THIS MONTH	17	65	44	31	92	77

$$\begin{array}{cc} \begin{array}{r}\text{Last Month}\\ \text{This Month}\end{array} & \begin{bmatrix} 12 & 58 & 37 \\ 17 & 65 & 44 \end{bmatrix} \\ & \begin{array}{ccc}\text{Pink} & \text{Gold} & \text{White}\end{array} \end{array} \qquad \begin{array}{cc} \begin{array}{r}\text{Last Month}\\ \text{This Month}\end{array} & \begin{bmatrix} 23 & 83 & 62 \\ 31 & 92 & 77 \end{bmatrix} \\ & \begin{array}{ccc}\text{Pink} & \text{Gold} & \text{White}\end{array} \end{array}$$

To determine the total number of sales for both models together, add the two matrices by adding the corresponding entries from the two addend matrices together. Place the sum of the two entries in the same row and column in the sum matrix.

$$\begin{bmatrix} 12 & 58 & 37 \\ 17 & 65 & 44 \end{bmatrix} + \begin{bmatrix} 23 & 83 & 62 \\ 31 & 92 & 77 \end{bmatrix} = \begin{bmatrix} 12+23 & 58+83 & 37+62 \\ 17+31 & 65+92 & 44+77 \end{bmatrix} = \begin{bmatrix} 35 & 141 & 99 \\ 48 & 157 & 121 \end{bmatrix}$$

SUBTRACTING MATRICES

As with addition, to subtract two matrices, you must first make sure that the matrices are the same size. The subtrahend matrix and the minuend matrix must have the same number of rows and the same number of columns.

In general, to subtract matrix A from matrix B, you subtract each pair of corresponding elements.

$$\begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \\ b_{3,1} & b_{3,2} \end{bmatrix} - \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{bmatrix} = \begin{bmatrix} b_{1,1}-a_{1,1} & b_{1,2}-a_{1,2} \\ b_{2,1}-a_{2,1} & b_{2,2}-a_{2,2} \\ b_{3,1}-a_{3,1} & b_{3,2}-a_{3,2} \end{bmatrix}$$

Using the mobile phone data from before, you can also calculate how many more Model 5c phones were sold than Model 5 phones. Subtract the Model 5 matrix from the Model 5c matrix. Place the difference between the two entries in the same row and column in the difference matrix.

In subtraction, the subtrahend is the number being subtracted from the minuend.
Minuend – **Subtrahend** = Difference

$$\begin{bmatrix} 23 & 83 & 62 \\ 31 & 92 & 77 \end{bmatrix} - \begin{bmatrix} 12 & 58 & 37 \\ 17 & 65 & 44 \end{bmatrix} = \begin{bmatrix} 23-12 & 83-58 & 62-37 \\ 31-17 & 92-65 & 77-44 \end{bmatrix} = \begin{bmatrix} 11 & 25 & 25 \\ 14 & 27 & 33 \end{bmatrix}$$

ADDING AND SUBTRACTING MATRICES

In order to add or subtract two matrices, they must have the same dimensions (number of rows and number of columns).

- As with real numbers, addition of matrices is commutative, but subtraction of matrices is not.
- To add two matrices, add the corresponding entries from the two addend matrices. Place the sum in the corresponding entry of the sum matrix.
- To subtract two matrices, subtract the corresponding entries from the two matrices being subtracted, paying attention to the order of the two matrices. Place the difference in the corresponding entry of the difference matrix.
- Technology can be used to add or subtract large matrices.

EXAMPLE 1

The Right Around the Corner Entertainment Center carries new and used electronic games, videos, and audio CDs. In Table 1, the company's end-of-year inventory is shown by new and used items. The company received a delivery of additional items, shown in Table 2. Construct a matrix for each table and then find the company's new inventory by taking the sum of the two matrices and displaying the results in the sum matrix.

TABLE 1

	YEAR END INVENTORY	
	NEW	USED
GAMES	408	223
VIDEOS	383	505
AUDIO CDS	178	244

TABLE 2

	NEW DELIVERY	
	NEW	USED
GAMES	120	90
VIDEOS	125	135
AUDIO CDS	110	115

STEP 1 Construct a matrix for Table 1 and 2.

The way that the table is set up suggests how we can organize the matrices, with the new and used categories for the columns and the types of products to categorize the rows.

Matrix M_1

$$\begin{bmatrix} 408 & 223 \\ 383 & 505 \\ 178 & 244 \end{bmatrix}$$

Matrix M_2

$$\begin{bmatrix} 120 & 90 \\ 125 & 135 \\ 110 & 115 \end{bmatrix}$$

STEP 2 Add M_1 and M_2 to find and display the new inventory.

$$S = \begin{bmatrix} 408+120 & 223+90 \\ 383+125 & 505+135 \\ 178+110 & 244+115 \end{bmatrix} = \begin{bmatrix} 528 & 313 \\ 508 & 640 \\ 288 & 359 \end{bmatrix}$$

The matrix $\begin{bmatrix} 528 & 313 \\ 508 & 640 \\ 288 & 359 \end{bmatrix}$ displays the store's new inventory.

YOU TRY IT! #1

Given the tables of coffee sales in thousands of dollars, place the data into matrices A_1 and A_2. Add the matrices, $A_1 + A_2$, to find A_3, a matrix with the sales for the first half of the year. Verify that the addition of matrices is commutative by verifying that $A_2 + A_1 = A_1 + A_2$.

TABLE 1

	FIRST QUARTER SALES	
	LATTES	MOCHAS
TALL	28	38
GRANDE	35	51
VENTI	42	36

TABLE 2

	SECOND QUARTER SALES	
	LATTES	MOCHAS
TALL	12	19
GRANDE	22	32
VENTI	24	27

EXAMPLE 2

The extreme high and low temperatures for the winter months of 2015 in a town on the Canadian border are given in the list as well as the average highs and lows for the area. Construct a 2 × 3 matrix for each list, use the matrices to compare the extreme temperatures from 2015 to the average temperatures, and interpret the results.

2015 extreme temperatures: December, highest 63° and lowest 17°; January, highest 59° and lowest -4°; February, highest 28° and lowest –6°

Average temperatures from 1978 to 2014: December, high 38° and low 18°; January, high 36° and low 17°; February, high 38° and low 20°.

STEP 1 Construct matrix *E* for the 2015 extreme temperatures and matrix *A* for the average temperatures for the winter months.

$$E = \begin{array}{c} \begin{array}{ccc} \text{Dec} & \text{Jan} & \text{Feb} \end{array} \\ \begin{bmatrix} 63 & 59 & 28 \\ 17 & -4 & -6 \end{bmatrix} \end{array} \quad A = \begin{array}{c} \begin{array}{ccc} \text{Dec} & \text{Jan} & \text{Feb} \end{array} \\ \begin{bmatrix} 38 & 36 & 38 \\ 18 & 17 & 20 \end{bmatrix} \end{array}$$

Step 2 Compare the extreme temperatures to the average temperatures for each month by subtracting matrix *A* from matrix *E* to create matrix *D*.

$$D = \begin{bmatrix} 63 - 38 & 59 - 36 & 28 - 38 \\ 17 - 18 & -4 - 17 & -6 - 20 \end{bmatrix} = \begin{bmatrix} 25 & 23 & -10 \\ -1 & -21 & -26 \end{bmatrix}$$

STEP 3 **Interpret the results.**

The differences have specific meaning depending on the sign. In $D_{1,1}$, the difference 25 means that the extreme high temperature was 25° above the average high temperature for December. In $D_{2,3}$, the difference -26 shows that the extreme low temperature was 26° below the average low temperature for February.

YOU TRY IT! #2

Place the data from the tables in a matrix for each. Find how many more students are enrolled in athletics for each grade level by gender by subtracting the matrices and showing the differences in the matrix. Determine whether the subtraction of matrices is commutative.

TABLE 1	ATHLETICS ENROLLMENT	
	GIRLS	BOYS
9TH GR.	74	85
10TH GR.	66	78
11TH GR.	57	73
12TH GR.	43	68

TABLE 2	BAND ENROLLMENT	
	GIRLS	BOYS
9TH GR.	52	58
10TH GR.	48	51
11TH GR.	46	46
12TH GR.	39	35

EXAMPLE 3

Brothers Fernando and Ramon have a friendly contest about who can make more money during the summer. The tables show how much each of them has earned at the end of each summer month by how much they have deposited in their checking and savings accounts. Make a matrix with the data from each table, and subtract the matrices to compare their earnings. Use the difference matrix to compare their overall earnings to determine who wins the contest.

TABLE 1	FERNANDO'S EARNINGS	
	CHECKING	SAVINGS
JUNE	$425	$50
JULY	$375	$40
AUG.	$450	$30

TABLE 2	RAMON'S EARNINGS	
	CHECKING	SAVINGS
JUNE	$560	$30
JULY	$515	$40
AUG.	$410	$20

STEP 1 Place the data from the tables into matrices.

$$F = \begin{bmatrix} 425 & 50 \\ 375 & 40 \\ 450 & 30 \end{bmatrix} \quad R = \begin{bmatrix} 560 & 30 \\ 515 & 40 \\ 410 & 20 \end{bmatrix}$$

STEP 2 Subtract the matrices, $R - F$, to compare Ramon's earnings to Fernando's.

$$D = \begin{bmatrix} 560 - 425 & 30 - 50 \\ 515 - 375 & 40 - 40 \\ 410 - 450 & 20 - 30 \end{bmatrix} = \begin{bmatrix} 135 & -20 \\ 140 & 0 \\ -40 & -10 \end{bmatrix}$$

STEP 3 Matrix D shows the comparison of Ramon's earnings to Fernando's. In June, he has earned \$115 more than Fernando because he has \$135 more in checking and \$20 less in savings. In July, Ramon has earned \$140 more if his deposits in checking and savings are combined. In August, Ramon has deposited a total of \$50 less than Fernando. So, overall, Ramon is \$115 + \$140 – \$50, or \$205 ahead of Fernando, so Ramon wins their contest.

YOU TRY IT! #3

Fernando and Ramon's parents require them to combine their summer earnings to help buy a car to share. The parents say they will match what the brothers make for each month that they deposit \$1,000 or more together. Use Matrices F and R to determine Matrix S, the sum of their monthly earnings and to see if they can take their parents up on their offer for any of the months.

$$F = \begin{bmatrix} 425 & 50 \\ 375 & 40 \\ 450 & 30 \end{bmatrix} \quad R = \begin{bmatrix} 560 & 30 \\ 515 & 40 \\ 410 & 20 \end{bmatrix}$$

PRACTICE/HOMEWORK

Use the scenario below to answer questions 1 – 5.

STATISTICS

Mr. Alvarez and Ms. Bento each recorded the grades of their students in their first period classes. The table below shows the number of students who received each letter grade during each of the first three grading periods. Both teachers have 40 students in each class.

MR. ALVAREZ

	1ST GRADING PERIOD	2ND GRADING PERIOD	3RD GRADING PERIOD
A	10	8	12
B	12	15	10
C	8	7	11
D	6	7	6
F	4	3	1

MS. BENTO

	1ST GRADING PERIOD	2ND GRADING PERIOD	3RD GRADING PERIOD
A	4	7	9
B	21	16	19
C	10	14	10
D	4	1	2
F	1	2	0

1. Construct matrix A to represent Mr. Alvarez' grades and matrix B to represent Ms. Bento's grades.

2. What is the combined number of students receiving an A, B, C, D, and F for the two teachers' classes? Express your answer as matrix T.

3. Using matrix T, what is the value of $T_{4,3}$ and what does it represent in the situation?

4. Find the difference between Mr. Alvarez's and Ms. Bento's grades $[A]-[B]=[D]$.

5. What is the value of $D_{3,2}$ and what does it represent in the situation?

Use the scenario below to answer questions 6 – 10.

STATISTICS

Rodrigo is preparing a report on internet usage in various countries. He gathers data for two different clusters of countries. The tables below shows the data he has collected on internet usage for the years 2000 and 2015. The values in the table are expressed in millions.

	CLUSTER A	
	2000	2015
CHINA	22.5	674
INDIA	5	375
JAPAN	47.1	114.9

	CLUSTER B	
	2000	2015
UNITED STATES	95.3	280.7
BRAZIL	5	117.7
RUSSIA	3.1	103.1

Source: www.internetworldstats.com/top20

6. Construct matrix A to represent the data from Cluster A and matrix B to represent the data from Cluster B.

7. What are the values of $A_{1,2}$ and $B_{3,1}$ and what do they represent in the situation?

8. Rodrigo decides to combine the data in the two clusters. Combine Cluster A with Cluster B and express your answer as matrix C.

9. Using matrix C, what is the value of $C_{2,2}$ and what does it represent in the situation?

10. Rodrigo wanted to find the difference in the data between the two clusters. He subtracted the data in Cluster A from the data in Cluster B and created matrix D, which is shown below.

$$D = \begin{bmatrix} 72.8 & -393.3 \\ 0 & -257.3 \\ -44 & -11.8 \end{bmatrix}$$

Which of the following conclusions can Rodrigo make about the data using matrix D? Select all conclusions that can be made.

A. Most of Cluster A has more internet users than Cluster B.
B. Most of Cluster B has more internet users than Cluster A.
C. In the year 2015, all of the countries in Cluster B had more internet users than all of the countries in Cluster A.
D. In the year 2015, all of the countries in Cluster A had more internet users than all of the countries in Cluster B.

Use the scenario below to answer questions 11 – 15.

STATISTICS

On a day in December, two movie theaters recorded the number and type of tickets they sold for four different movies. The data is shown in the tables below.

AVALON THEATER

	ADULT	STUDENT	CHILD	SENIOR
STAR WARS THE FORCE AWAKENS	650	525	460	275
THE BIG SHORT	309	188	17	198
CONCUSSION	485	301	22	214
THE REVENANT	517	480	73	165
TOTAL	**1961**	**1494**	**572**	**852**

BIJOU THEATER

	ADULT	STUDENT	CHILD	SENIOR
STAR WARS THE FORCE AWAKENS	704	611	398	149
THE BIG SHORT	436	102	10	211
CONCUSSION	517	376	37	155
THE REVENANT	684	310	82	227
TOTAL	**2341**	**1399**	**527**	**742**

11. Construct matrix A to represent the ticket sales at Avalon Theater and matrix B to represent the ticket sales at Bijou Theater.

12. The same company owns both theaters and want to see their combined ticket sales. Express their combined ticket sales as matrix C.

13. Using matrix C, what is the value of $C_{2,3}$ and $C_{5,4}$ and what do they represent in the situation?

14. The company subtracted the data from the Bijou Theater from the data from the Avalon Theater. Find the difference in the two data sets and express the difference as matrix D.

15. Based on the data shown in matrix D, which theater sold the most adult tickets and by how much?

Use the matrices below to answer questions 16 – 20.

$$A = \begin{bmatrix} 2 & 7 & -5 \\ 9 & 0 & 4 \\ -4 & 6 & 11 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -7 & 12 \\ 3 & -6 & -8 \\ 15 & 2 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 18 & -9 & -3 \\ 1 & 5 & -5 \\ 4 & 14 & -7 \end{bmatrix}$$

Perform the indicated operations. Express your answer as a matrix.

16. $[B] + [C]$

17. $[C] - [A]$

18. $[B] + [A]$

19. $[A] - [C]$

20. $[A] + [C]$

6.3 Scalar Multiplication of Matrices

FOCUSING QUESTION How do I multiply a matrix by a scalar quantity?

LEARNING OUTCOMES

- I can multiply a data set that is represented in a matrix by a scalar.
- I can apply mathematics to solve problems arising in the workplace.

ENGAGE

Merrily was shopping for gifts for her family. She purchased a video game that cost \$39.95, a purse that cost \$19.95, and 3 T-shirts that cost \$14.95 each. She had a coupon for 20% off her entire purchase and she had to pay 8.25% sales tax on her subtotal, after the discount. What was the final cost of Merrily's purchases, including the discount and sales tax?

EXPLORE

Nicholas manages a popcorn store at a local mall. The store sells popcorn mixes in several flavors. Each flavor is available in a small, medium, or large package. Nicholas made a table, which is shown, with the prices for the five most popular flavors of popcorn.

ITEM	SMALL	MEDIUM	LARGE
CHEDDAR CHAMPION	\$8.50	\$10.50	\$12.50
PEPPERMINT BARK	\$7.95	\$9.95	\$11.95
CHOCOLATE CRUNCH	\$8.25	\$10.25	\$12.25
PEANUT BUTTER DELIGHT	\$9.15	\$11.15	\$13.15
MIDNIGHT MOCHA	\$8.90	\$10.90	\$12.90

1. Construct matrix A to display the prices.

2. Where Nicholas lives, the popcorn sales are taxable and the sales tax rate is 6.5%. Write an expression that you can use to calculate the final price of a package of popcorn, including sales tax. Let x represent the price of one package of popcorn. Simplify your expression completely.

3. Construct matrix P showing the price of each package of popcorn including sales tax. Use either paper and pencil or technology to perform the computations. Round to the nearest cent if necessary.

4. Use the matrices to identify the values of $a_{3,1}$ and $p_{3,1}$.

5. Use the matrices to identify the values of $a_{4,2}$ and $p_{4,2}$.

6. What is the ratio of $\frac{p_{3,1}}{a_{3,1}}$? The ratio of $\frac{p_{4,2}}{a_{4,2}}$?

7. What do these ratios represent?

8. How are the entries in matrix P, $p_{R,C}$, related to their corresponding entries in matrix A, $a_{R,C}$?

9. Use graphing technology such as a graphing calculator or app to enter the data from matrix A. Remember that matrices are defined by the number of rows by the number of columns.

10. Use matrix operations to multiply matrix A by 1.065. On some devices, this will appear on the home screen as $1.065 \times [A]$ and the product will appear as a new matrix. How does this matrix compare to matrix P, the one you calculated in a previous question?

11. For online orders that Nicholas ships in-state, customers must pay both sales tax and a 9% shipping charge. What number would you multiply by matrix A, the original prices of packages of popcorn without sales tax, in order to generate matrix S, the cost of each package of popcorn with sales tax and shipping included?

12. Use your graphing technology to generate matrix S from the scale factor you identified and the entries in matrix A. Round to the nearest cent if necessary.

13. How are the entries in matrix S, $s_{R,C}$, related to their corresponding entries in matrix A, $a_{R,C}$ and the scale factor representing the combined sales tax and shipping rates?

REFLECT

- **When you multiply a matrix by a scale factor, what happens to the values of each entry in the matrix?**

- **How does multiplying a matrix by a number with paper and pencil compare to multiplying a matrix by a number using technology?**

EXPLAIN

ELPS ACTIVITY

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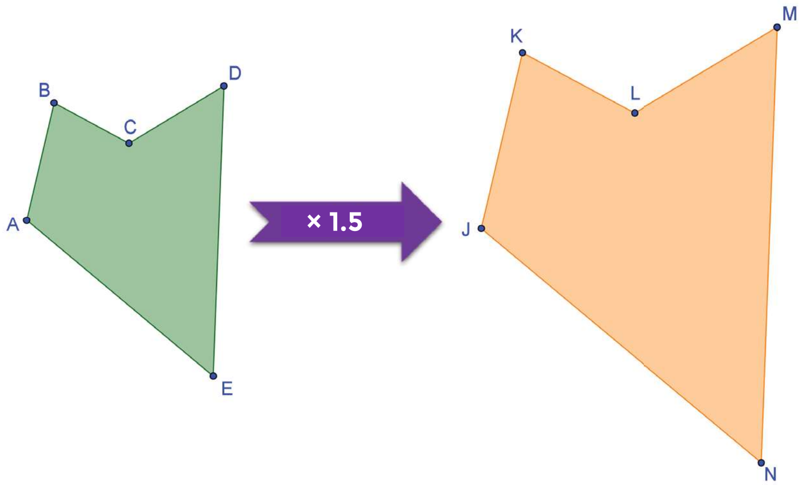
Read the passage below along with a peer. Ask each other questions like those below to enhance and confirm your understanding.

- What does this remind you of?
- How could you say that differently?
- How is the operation with the matrix similar to the operation with numbers?

If you encounter words that you and your peer do not know, ask your teacher for additional support.

In mathematics, a **scalar** is a real number that is used to indicate the size, or scale, of something. Scalars can also be used as scale factors that indicate a proportional enlargement or reduction of an object. When you multiply the side lengths of a polygon by the same scale factor, or scalar, and keep the angle measures the same, you generate a similar figure.



Likewise, when you multiply a matrix by a scalar, you are proportionally enlarging or reducing the values of the entries in the matrix. The dimensions of the matrix do not change. Only the values of the entries in the matrix are changed proportionally.

MULTIPLYING A MATRIX BY A SCALAR

In general, to multiply matrix A by a scalar, k, you multiply each entry of matrix A by k.

$$k\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{bmatrix} = \begin{bmatrix} ka_{1,1} & ka_{1,2} \\ ka_{2,1} & ka_{2,2} \\ ka_{3,1} & ka_{3,2} \end{bmatrix}$$

For example, the table shows the price of three different size packages of popcorn for each of five flavors in Nicholas's popcorn store.

ITEM	SMALL	MEDIUM	LARGE
CHEDDAR CHAMPION	$8.50	$10.50	$12.50
PEPPERMINT BARK	$7.95	$9.95	$11.95
CHOCOLATE CRUNCH	$8.25	$10.25	$12.25
PEANUT BUTTER DELIGHT	$9.15	$11.15	$13.15
MIDNIGHT MOCHA	$8.90	$10.90	$12.90

The data can be entered into a matrix.

$$\begin{bmatrix} 8.50 & 10.50 & 12.50 \\ 7.95 & 9.95 & 11.95 \\ 8.25 & 10.25 & 12.25 \\ 9.15 & 11.15 & 13.15 \\ 8.90 & 10.90 & 12.90 \end{bmatrix}$$

If Nicholas decides to have a sale where all popcorn is 20% off, then he can multiply the matrix by the scalar 80% = 0.8 in order to calculate the discounted prices.

$$B = 0.8A = \begin{bmatrix} 0.8(8.50) & 0.8(10.50) & 0.8(12.50) \\ 0.8(7.95) & 0.8(9.95) & 0.8(11.95) \\ 0.8(8.25) & 0.8(10.25) & 0.8(12.25) \\ 0.8(9.15) & 0.8(11.15) & 0.8(13.15) \\ 0.8(8.90) & 0.8(10.90) & 0.8(12.90) \end{bmatrix} = \begin{bmatrix} 6.80 & 8.40 & 10.00 \\ 6.36 & 7.96 & 9.56 \\ 6.60 & 8.20 & 9.80 \\ 7.32 & 8.92 & 10.52 \\ 7.12 & 8.72 & 10.32 \end{bmatrix}$$

Nicholas can now use the rows and columns in the product matrix B to determine the sale price of each bag of popcorn. For example, $b_{4,1}$ = $7.32, which means that a small bag of Peanut Butter Delight will cost $7.32 during the 20% off sale.

DISTRIBUTIVE PROPERTY WITH MATRICES

With real numbers, and algebraic expressions containing real numbers, the distributive property states that:

$$a(b + c) = ab + ac$$

$$a(b - c) = ab - ac$$

In other words, if you have a number that you're multiplying by a sum (or difference) of two numbers, you can multiply the number by each addend (or subtrahend and minuend) then add (or subtract) and get the same answer.

Let's see if this property works with matrices.

Suppose that a group of people with 4 adults, 4 kids, and 4 seniors plans to have a meal together and see a movie. The restaurant where they will be dining offers *prix fixe* meals, which is an appetizer, main course, and dessert for a set price.

Restaurant Prices

- Lunch: $10 for kids, $15 for adults, $12 for seniors
- Dinner: $12 for kids, $25 for adults, $20 for seniors

Movie Prices

- Matinee: $5 for kids, $8 for adults, $6 for seniors
- Evening: $7 for kids, $10 for adults, $7 for seniors

Each set of prices can be placed into a 2×3 matrix.

Matrix *R*: Restaurant Prices

$$\begin{array}{cc} \begin{array}{ccc} K & A & S \end{array} & \\ \begin{bmatrix} 10 & 15 & 12 \\ 12 & 25 & 20 \end{bmatrix} & \begin{array}{l} \text{Lunch} \\ \text{Dinner} \end{array} \end{array}$$

Matrix *M*: Movie Prices

$$\begin{array}{cc} \begin{array}{ccc} K & A & S \end{array} & \\ \begin{bmatrix} 5 & 8 & 6 \\ 7 & 10 & 7 \end{bmatrix} & \begin{array}{l} \text{Matinee} \\ \text{Evening} \end{array} \end{array}$$

The combined cost of the meal and movie for 4 adults, 4 kids, and 4 seniors can be represented with the matrix equation shown. This matrix expression can be simplified using the order of operations. Then, we can use the distributive property to see if the answer is the same.

$$4\left(\begin{bmatrix} 10 & 15 & 12 \\ 12 & 25 & 20 \end{bmatrix} + \begin{bmatrix} 5 & 8 & 6 \\ 7 & 10 & 7 \end{bmatrix}\right)$$

Method 1: According to the order of operations, add the matrices inside the grouping symbols (parentheses) first. Then, multiply the sum by 4.

$$\underbrace{4\left(\begin{bmatrix} 10 & 15 & 12 \\ 12 & 25 & 20 \end{bmatrix} + \begin{bmatrix} 5 & 8 & 6 \\ 7 & 10 & 7 \end{bmatrix}\right)}_{\text{4(restaurant + movie)}} = \underbrace{4\left(\begin{bmatrix} 15 & 23 & 18 \\ 19 & 35 & 27 \end{bmatrix}\right)}_{\text{4(combined price)}} = \underbrace{\begin{bmatrix} 60 & 92 & 72 \\ 76 & 140 & 108 \end{bmatrix}}_{\text{final price}}$$

Method 2: Apply the distributive property.

$$\underbrace{4\left(\begin{bmatrix} 10 & 15 & 12 \\ 12 & 25 & 20 \end{bmatrix} + \begin{bmatrix} 5 & 8 & 6 \\ 7 & 10 & 7 \end{bmatrix}\right)}_{\text{4(restaurant + movie)}} = \underbrace{4\left(\begin{bmatrix} 10 & 15 & 12 \\ 12 & 25 & 20 \end{bmatrix}\right)}_{\text{4(restaurant)}} + \underbrace{4\left(\begin{bmatrix} 5 & 8 & 6 \\ 7 & 10 & 7 \end{bmatrix}\right)}_{\text{4(movie)}}$$

$$= \underbrace{\begin{bmatrix} 40 & 60 & 48 \\ 48 & 100 & 80 \end{bmatrix} + \begin{bmatrix} 20 & 32 & 24 \\ 28 & 40 & 28 \end{bmatrix}}_{\text{total restaurant + total movie}} = \underbrace{\begin{bmatrix} 60 & 92 & 72 \\ 76 & 140 & 108 \end{bmatrix}}_{\text{total price}}$$

The distributive property appears to work for matrices. It can be algebraically shown that the distributive property will work for both scalar multiplication over addition and scalar multiplication over subtraction with matrices.

SCALAR MULTIPLICATION OF MATRICES

A matrix can be multiplied by a scalar quantity.

- If the scalar is *k*, multiply each entry in the matrix by *k*. Place the product in the corresponding entry of the product matrix.
- The distributive property of scalar multiplication over addition applies to matrices.
- The distributive property of scalar multiplication over subtraction applies to matrices.

EXAMPLE 1

Multiply matrix M by the scalar $\frac{7}{8}$.

$$M = \begin{bmatrix} 16 & -24 \\ -40 & 72 \end{bmatrix}$$

STEP 1 **Decide whether to use the fractional or decimal form of $\frac{7}{8}$.**

Notice that each of the entries in matrix M is a multiple of 8. Choosing the fractional form would be easier and can be accomplished without technology.

STEP 2 **Multiply each entry in matrix *M* by $\frac{7}{8}$ to create matrix *S*.**

$$S = \frac{7}{8}M = \frac{7}{8} \cdot \begin{bmatrix} 16 & -24 \\ -40 & 72 \end{bmatrix} = \begin{bmatrix} \frac{7}{8} \cdot 16 & \frac{7}{8} \cdot -24 \\ \frac{7}{8} \cdot -40 & \frac{7}{8} \cdot 72 \end{bmatrix} = \begin{bmatrix} 14 & -21 \\ -35 & 63 \end{bmatrix}$$

STEP 3 **How are the entries in matrix *S* related to their corresponding entries in matrix *M*?**

Choose $s_{2,1}$ and $m_{2,1}$ to compare: The ratio of –35 to –40 is 7 to 8.

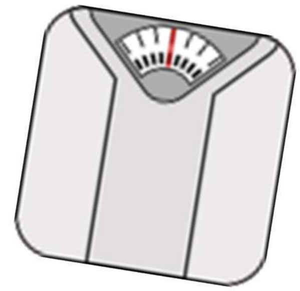
YOU TRY IT! #1

Using the scalar –0.189, multiply matrix M to create matrix S. Choosing technology might be helpful with the multiplication by a decimal scalar.

$$M = \begin{bmatrix} 8 & 10 & 12 \\ -7 & -9 & 11 \\ 8 & 15 & 13 \\ 9 & 19 & 14 \\ -6 & 16 & 17 \end{bmatrix}$$

EXAMPLE 2

A group of three girls and three boys decided to lose weight and get in shape. They set a goal of losing a fourth of their body weight over the course of a year. Matrix W shows their initial weights, with the boys' weights in row 1 and the girls' weights in row 2. How much weight must each student lose to meet their target goal? What are some other uses of the matrix here?



$$W = \begin{bmatrix} 228 & 198 & 229 \\ 179 & 199 & 182 \end{bmatrix}$$

STEP 1 **Multiply matrix *W* by the scalar −0.25 to create matrix *L*, the amount of weight each intends to lose.**

$$\text{Matrix } L = -0.25\,W = -0.25\begin{bmatrix} 228 & 198 & 229 \\ 179 & 199 & 182 \end{bmatrix} = \begin{bmatrix} -57 & -49.5 & -57.25 \\ -44.75 & -49.75 & -45.5 \end{bmatrix}$$

STEP 2 **Interpret the results in Matrix *L* and describe other uses of the matrix.**

The negative sign indicates a loss or negative change in weight. The more the initial weight, the more weight will be lost. The girl whose weight is entered in $w_{2,3}$ as 182 pounds has a goal of losing 45.5 pounds.

Finding their target weight would be easy with technology by adding matrix L from matrix W. Finding their midway weight loss could be found by multiplying matrix W by the scalar, –0.125. So once the matrix is constructed in technology, there are many purposes that can be accomplished easily.

YOU TRY IT! #2

Multiply matrix Q, containing the menu prices of three sizes of sandwiches and three sizes of drinks, by the scalar 1.0725 to determine matrix S, the cost of various sandwiches and drinks with sales tax. You might want to choose to do the multiplication with the aid of technology. Businesses usually round up charges to the next cent.

$$Q = \begin{bmatrix} 5.28 & 6.99 & 7.29 \\ 0.79 & 0.99 & 1.29 \end{bmatrix}$$

EXAMPLE 3

The PizzaPlenty pizza delivery company has two locations, one on the north side of town and one on the south side. The company tracks their sales and profits using matrices. The average profit per pizza this past year was \$1.95 per pizza at both locations. Their biggest sales are on Halloween and Super Bowl Sunday. Using the matrices for the number of pizzas sold at the north (row 1) and south (row 2) locations on each of the days, find out how much more profit is generated on Super Bowl Sunday than on Halloween. What information, if any, can be found using the distributive property?

Matrix H: Halloween

$$H = \begin{array}{ccc} \text{Cheese} & \text{Supreme} & \text{Sausage} \\ \end{array}$$
$$H = \begin{bmatrix} 48 & 55 & 72 \\ 114 & 89 & 93 \end{bmatrix}$$

Matrix H: Super Bowl

$$S = \begin{array}{ccc} \text{Cheese} & \text{Supreme} & \text{Sausage} \\ \end{array}$$
$$S = \begin{bmatrix} 70 & 68 & 85 \\ 129 & 110 & 108 \end{bmatrix}$$

STEP 1 **Use the verbs in the situation to determine the operation that will determine how many more pizzas of each kind and at each location were sold on Super Bowl Sunday than on Halloween.**

When a question asks about the difference between two numbers, you should subtract matrix H from matrix S.

STEP 2 **Use what you know about profit and sales to determine the operation that will tell you how much profit was made on Super Bowl Sunday or on Halloween.**

You know the profit for one pizza is, \$1.95. To determine the profit on a set of pizzas, scale up the profit for one pizza. Scaling is one meaning of multiplication, so you would need to multiply matrix S by the scalar for profit, \$1.95. That will give you the total profit from each of the kinds of pizzas from each location for Super Bowl Sunday. For Halloween, multiply matrix H by \$1.95.

STEP 3 **Determine the operation you need to use to calculate how much more profit was made on Super Bowl Sunday than on Halloween.**

The difference in profit for each event can be found with subtraction. You could subtract 1.95[H] from 1.95[S], giving you matrix P_1, which uses the distributive property. Or you could subtract matrix H from matrix S and then multiply the difference by 1.95, giving you matrix P_2.

$$P_1 = 1.95\begin{bmatrix} 70 & 68 & 85 \\ 129 & 110 & 108 \end{bmatrix} - 1.95\begin{bmatrix} 48 & 55 & 72 \\ 114 & 89 & 93 \end{bmatrix}$$

$$P_2 = 1.95\left(\begin{bmatrix} 70 & 68 & 85 \\ 129 & 110 & 108 \end{bmatrix} - \begin{bmatrix} 48 & 55 & 72 \\ 114 & 89 & 93 \end{bmatrix}\right)$$

STEP 4 **Determine whether the profit matrices P_1 and P_2 are equal and explain the information you get from each method.**

$$P_1 = \begin{bmatrix} 136.50 & 132.60 & 165.75 \\ 251.55 & 214.50 & 210.60 \end{bmatrix} - \begin{bmatrix} 93.60 & 107.25 & 140.40 \\ 222.30 & 173.55 & 181.35 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 42.90 & 25.35 & 25.35 \\ 29.25 & 40.95 & 29.25 \end{bmatrix}$$

$$P_2 = 1.95\begin{bmatrix} 22 & 13 & 13 \\ 15 & 21 & 15 \end{bmatrix} = \begin{bmatrix} 42.90 & 25.35 & 25.35 \\ 29.25 & 40.95 & 29.25 \end{bmatrix}$$

The results in matrix P_1 and in matrix P_2 are the same. If you use the first method, $1.95S - 1.95H$, you can see the profit from each type of pizza at each location. If you use the second method, $1.95(S - H)$, you can see how many more pizzas of each type were sold at each location.

YOU TRY IT! #3

To apply the sales tax to a purchase price, you find the tax by multiplying the purchase price by the percent of the tax and then add the tax to the purchase price to find the total cost. You can also add the percent of the tax to the percent of the purchase price, which is 100%, and then multiply the purchase price by the combined percent. To verify this, find the total cost with a 7.5% sales tax included on the purchase prices of a pair of jeans, a shirt, and a belt shown in matrix P. Employ the method without using the distributive property (matrix T_1) and with using it (matrix T_2).

$$P = \begin{bmatrix} 45.79 & 23.59 & 10.99 \end{bmatrix}$$

PRACTICE/HOMEWORK

Use the following matrices to answer questions 1 – 8.

$$A = \begin{bmatrix} 3 & -1 & 5 \\ 2 & 1 & -4 \\ -6 & 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -6 & -8 \\ -2 & 10 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -3 \\ 5 & -4 \\ -7 & 5 \end{bmatrix} \qquad D = \begin{bmatrix} 1.375 & 0.875 & -2.5 \\ -3.625 & 2.125 & -4.75 \end{bmatrix}$$

1. Using the scalar 3, multiply matrix A to create matrix M.

2. Using the scalar $-\frac{1}{2}$, multiply matrix B to create matrix N.

3. Using the scalar 1.085, multiply matrix C to create matrix P.

4. Using the scalar -2, multiply matrix D to create matrix R.

5. Using the scalar -1.65, multiply matrix A to create matrix S.

6. Using the scalar -0.8, multiply matrix B to create matrix T.

7. Using the scalar 3.64, multiply matrix C to create matrix X.

8. Using the scalar 4, multiply matrix D to create matrix Y.

Use the following scenario for questions 9 - 12.

FINANCE

Maria bought a jacket, a shirt, and a pair of jeans at two different stores. Matrix M shows the cost of the jacket, shirt, and jeans at store A on row 1 and store B on row 2.

$$M = \begin{bmatrix} 42.80 & 24.30 & 38.90 \\ 36.60 & 28.80 & 36.40 \end{bmatrix}$$

9. Both stores offer a discount of 40% off any purchase. Create matrix D to show the amount of discount on each item at each store.

10. Create matrix C to show the amount each item will cost Maria with the discount.

11. The sales tax Maria must pay is 8.25%. Create matrix S to show the amount of tax Maria must pay for each item.

12. Create matrix T to show the final cost of each item after the discount and sales tax.

Use the following scenario for questions 13 - 16.

FINANCE

Andrea and Tony work at a sporting goods store. Matrix N shows the number of hours that Andrea worked for four different weeks on row 1 and the number of hours that Tony worked for the same four weeks on row 2.

$$N = \begin{bmatrix} 34 & 40 & 26 & 38 \\ 24 & 38 & 40 & 36 \end{bmatrix}$$

13. Andrea and Tony earn \$16.50 an hour. Create matrix E to show the gross income for each of them for each week.

14. Andrea and Tony both pay 18% of their gross income in deductions. Create matrix D to show the amount of deductions that each of them will pay for each week.

15. Create matrix I to show Andrea's and Tony's net income for each week.

16. Andrea and Tony both save 6% of their net income each week. Create matrix S to show the amount of money each of them will same for each week.

Use the following scenario for questions 17 - 20.

GEOMETRY

Matrix T shows the lengths of the sides of three different triangles. The lengths of the sides or triangle A are on row 1, the lengths of the sides of triangle B are on row 2, and the lengths of the sides of triangle C are on row 3.

$$T = \begin{bmatrix} 6 & 8 & 10 \\ 4 & 7 & 9 \\ 12 & 12 & 14 \end{bmatrix}$$

17. The three triangles are dilated by a scale factor of 2. Create matrix W to show the lengths of the sides of the triangles after they are dilated.

18. The three original triangles are dilated by 75%. Create matrix X to show the lengths of the sides of the triangles after they are dilated.

19. The three original triangles are dilated by a scale factor of 3.5. Create matrix Y to show the lengths of the sides of the triangles after they are dilated.

20. The three original triangles are dilated by 60%. Create matrix Z to show the lengths of the sides of the triangles after they are dilated.

Chapter 6 Mid-Chapter Review

6.1 Representing Data in Matrices

Use matrix A to answer questions 1–4.

Matrix A shows the average revenue collected in U.S. dollars over the course of a year at the Premier Cinema for selling pizza, popcorn, and soda on Fridays and Saturdays.

AVERAGE CONCESSIONS REVENUE AT PREMIER CINEMA

$$A = \begin{array}{c} \\ \text{Friday} \\ \text{Saturday} \end{array} \begin{array}{c} \begin{array}{ccc} \text{pizza} & \text{popcorn} & \text{soda} \end{array} \\ \begin{bmatrix} 675 & 425 & 1{,}290 \\ 850 & 712 & 1{,}545 \end{bmatrix} \end{array}$$

1. What are the dimensions of Matrix A?

2. What is the value of entry $a_{2,1}$?

3. What is the value of entry $a_{1,3}$?

4. What is the meaning of the value of entry $a_{1,2}$?

Use the data in the table to answer questions 5 – 8.

POPULATION (IN MILLIONS) OF 3 LARGEST NORTH AMERICAN COUNTRIES

COUNTRY	1990	2000	2010
CANADA	27.8	30.8	34.0
MEXICO	85.6	102.8	118.6
UNITED STATES	249.6	282.2	309.3

Source: World Bank

5. Organize the population data into a matrix.

6. What are the dimensions of your matrix?

7. Describe the meaning of the entry in row 2, column 2.

8. How did the population of Canada compare to the population of the United States in the year 2010?

6.2 Adding and Subtracting Matrices

Use the matrices below to answer questions 9 – 12.

$$Q = \begin{bmatrix} -4 & 10 & 15 \\ 12 & -5 & 14 \\ 8 & -2 & 12 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 15 & 30 & 45 \end{bmatrix} \quad T = \begin{bmatrix} 20 & 16 & 12 \\ -5 & -4 & -3 \\ 4 & 5 & 6 \end{bmatrix}$$

Perform the indicated operations. Express your answer as a matrix.

9. $[Q] + [T]$ **10.** $[R] - [Q]$ **11.** $[Q] - [R]$

12. How do the results of $[R] - [Q]$ and $[Q] - [R]$ compare?

Use the data in the matrices below to answer questions 13 – 15.

During an election, eligible voters were polled to determine their preferences among three candidates. Matrix *A* shows the percent of all eligible voters who preferred each candidate in each of three months prior to the election. Matrix *U* shows the percent of unregistered voters who preferred each candidate..

$$A = \begin{array}{r} \\ \text{Smith} \\ \text{Jackson} \\ \text{Cardona} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Sept} & \text{Oct} & \text{Nov} \end{array} \\ \begin{bmatrix} 25 & 17 & 12 \\ 13 & 18 & 13 \\ 5 & 12 & 30 \end{bmatrix} \end{array} \quad U = \begin{array}{r} \\ \text{Smith} \\ \text{Jackson} \\ \text{Cardona} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Sept} & \text{Oct} & \text{Nov} \end{array} \\ \begin{bmatrix} 20 & 10 & 5 \\ 11 & 15 & 11 \\ 1 & 5 & 12 \end{bmatrix} \end{array}$$

13. Create matrix *R* consisting of registered voters who were included in the poll.

14. Which candidate was the most popular among all **eligible** voters in September?

15. Which candidate had the greatest change from September to November among **registered voters**?

6.3 Scalar Multiplication of Matrices

Use the following matrices to answer questions 16 – 19.

$$A = \begin{bmatrix} 10 & 8 & -5 \\ 6 & 4 & -3 \\ -14 & -12 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 20 & 45 & 90 \\ -15 & 30 & -75 \end{bmatrix} \quad C = \begin{bmatrix} 1.2 & -4.4 \\ 1.8 & 3.2 \\ -5 & 0.6 \end{bmatrix} \quad D = \begin{bmatrix} 6 & 8 & 12 \\ 4 & 3 & 2 \\ 15 & 10 & 5 \end{bmatrix}$$

16. Multiply [C] by a scalar of 5.

17. Multiply $[B]$ by a scalar of $-\frac{1}{5}$.

18. Find the result of $2[A] - [D]$

19. Find the result of $2[A] + 3[D]$

Use the following scenario to answer questions 20 – 22.

Jason's Candies sells trail mix, candied fruit, and mixed nuts in 3 different sizes. The matrix below shows the cost of each item.

$$\begin{array}{r c} & \begin{array}{ccc} \text{Sm} & \text{Med} & \text{Lrg} \end{array} \\ \begin{array}{r} \text{trail mix} \\ \text{fruit} \\ \text{nuts} \end{array} & \begin{bmatrix} 2.50 & 3.75 & 5.25 \\ 3.25 & 4.50 & 5.75 \\ 4.50 & 6.00 & 8.25 \end{bmatrix} \end{array}$$

20. If Jason must charge 7.5% sales tax on each item, create a matrix to represent the cost with the tax included. Round each value to the nearest penny.

21. If Jason offers a 20% discount on each item, create a matrix to represent the discounted cost before taxes.

22. After offering a 20% discount, Jason must then charge 7.5% tax. Create a matrix to represent the discounted cost plus taxes.

MULTIPLE CHOICE

23. Given matrix $M = \begin{bmatrix} 3 & -5 & 8 & 12 \\ 0 & 11 & 7 & 3 \\ -4 & 9 & -2 & 0 \end{bmatrix}$, which entries have a value greater than 10?

A. $m_{2,2}$ and $m_{4,1}$
B. $m_{3,4}$ and $m_{1,4}$
C. $m_{3,4}$ and $m_{2,2}$
D. $m_{2,2}$ and $m_{1,4}$

24. Matrix *A* and matrix *B* are shown below.

$$A = \begin{bmatrix} 9 & 12 & 15 \\ 1 & 4 & 7 \\ 3 & 4 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -14 & 24 \\ 9 & -18 & -24 \\ 15 & 12 & 20 \end{bmatrix}$$

Which matrix represents $[A] + [B]$?

A. $\begin{bmatrix} 9 & 21 & 30 \\ -13 & -14 & -17 \\ 27 & -20 & 25 \end{bmatrix}$ **B.** $\begin{bmatrix} 9 & -2 & 39 \\ 10 & -22 & -31 \\ 18 & 16 & 25 \end{bmatrix}$

C. $\begin{bmatrix} 9 & -2 & 39 \\ 10 & -14 & -17 \\ 18 & 16 & 25 \end{bmatrix}$ **D.** $\begin{bmatrix} 9 & 26 & 39 \\ 10 & 22 & 31 \\ 18 & 16 & 25 \end{bmatrix}$

25. The matrices below represent the attendance at two performances of a school play on both Saturday and Sunday by children (*c*), adults (*a*), and seniors (*s*).

SATURDAY ATTENDANCE

$$\begin{array}{r c} & \begin{array}{ccc} c & a & s \end{array} \\ \begin{array}{r} \text{matinee} \\ \text{evening} \end{array} & \begin{bmatrix} 27 & 75 & 40 \\ 12 & 120 & 22 \end{bmatrix} \end{array}$$

SUNDAY ATTENDANCE

$$\begin{array}{r c} & \begin{array}{ccc} c & a & s \end{array} \\ \begin{array}{r} \text{matinee} \\ \text{evening} \end{array} & \begin{bmatrix} 18 & 42 & 27 \\ 5 & 85 & 67 \end{bmatrix} \end{array}$$

If matrix *C* represents the combined attendance for both Saturday and Sunday, what would the entry $c_{2,1}$ represent?

A. A total of 322 adults attended the play.
B. A total of 117 adults attended the play during the matinees.
C. A total of 17 children attended the play during the evenings.
D. A total of 62 children attended the play.

26. Which of the following shows matrix *A* multiplied by a scalar of $\frac{3}{2}$? $A = \begin{bmatrix} 12 & 6 & 4 \\ 18 & 8 & 10 \end{bmatrix}$

A. $\frac{3}{2}[A] = \begin{bmatrix} 6 & 3 & 2 \\ 9 & 4 & 5 \end{bmatrix}$ **B.** $\frac{3}{2}[A] = \begin{bmatrix} 18 & 9 & 6 \\ 27 & 12 & 15 \end{bmatrix}$ **C.** $\frac{3}{2}[A] = \begin{bmatrix} 18 & 9 & 8 \\ 36 & 16 & 20 \end{bmatrix}$ **D.** $\frac{3}{2}[A] = \begin{bmatrix} 36 & 18 & 12 \\ 54 & 24 & 30 \end{bmatrix}$

27. Which of the following scalars was matrix *X* multiplied by to create matrix *Y*?

$$X = \begin{bmatrix} 36 & 60 & 12 \\ 24 & 72 & 18 \end{bmatrix} \qquad Y = \begin{bmatrix} 14.4 & 24 & 4.8 \\ 9.6 & 28.8 & 7.2 \end{bmatrix}$$

A. 40% **B.** $\frac{3}{5}$ **C.** $\frac{3}{4}$ **D.** 45%

Multiplying Matrices

6.4

FOCUSING QUESTION How do I multiply two matrices together?

LEARNING OUTCOMES

- I can multiply one matrix by another matrix.
- I can communicate mathematical reasoning about matrix multiplication using symbols, diagrams, and language.

ENGAGE

The table below represents the type and number of homes that were for sale in two neighborhoods in a recent month.

	HOUSES	CONDOMINIUMS	TOWNHOMES
OAK GROVE	25	20	30
FERN MEADOWS	20	15	35

If 20% of each type of home sells in both neighborhoods and no additional homes are placed for sale, how many homes will be for sale the next month?

EXPLORE

The tables below show the crop yields and average wholesale prices for different crops in Texas and Georgia during 2014.

	CORN (NUMBER OF BUSHELS)	COTTON (NUMBER OF BALES)	SOYBEANS (NUMBER OF BUSHELS)
TEXAS	294,520,000	6,203,000	5,198,000
GEORGIA	52,700,000	2,570,000	11,600,000

CORN	$4 PER BUSHEL
COTTON	$325.50 PER BALE
SOYBEANS	$10.50 PER BUSHEL

Source: U.S. Department of Agriculture

1. Create matrix Y to show the crop yields for both Texas and Georgia and matrix P to show the price for each crop. Identify the dimensions of each matrix.

2. Determine the amount of revenue generated for each crop (corn, cotton, and soybeans) for each state (Texas and Georgia). Explain your reasoning using mathematically-appropriate language.

3. Determine the total amount of revenue generated from the sale of these crops for both Texas and Georgia. Explain your reasoning using mathematically-appropriate language.

4. Create matrix T to show the total revenue generated from the sale of all crops for both states. Identify the dimensions of this matrix.

5. Compare the number of rows in the total matrix, T, to the number of rows in the crop yield matrix, Y. What do you notice?

6. Compare the number of columns in the total matrix, T, to the number of columns in the price matrix, P. What do you notice?

7. Use graphing technology such as a graphing calculator or app to enter the data from matrix Y and matrix P. Remember that matrices are defined by the number of rows by the number of columns.

8. Use matrix operations to multiply matrix Y by matrix P, $[Y] \times [P]$. (*Note: If you used other matrix names in your graphing technology such as matrix A and matrix B, you will use those matrix names in the calculation.*) How does this matrix compare to matrix T, the one you calculated in a previous question?

9. Test to see if matrix multiplication is commutative by reversing the order of the two factor matrices. In other words, use your graphing technology to multiply matrix P by matrix Y, or $[P] \times [Y]$. What do you notice?

REFLECT

- **When you multiply a matrix by another matrix, how do you multiply the entries in each matrix in order to determine the number to place in the product matrix?**

- **Is matrix multiplication commutative? Why or why not?**

EXPLAIN

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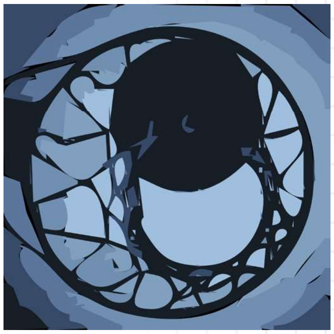
Previously, you multiplied a matrix by a scalar quantity. In doing so, you multiplied the scalar by each entry in the matrix. The product matrix has the same dimensions as the original factor matrix.

When you multiply a matrix by another matrix, you have to take pairs of entries from each factor matrix into consideration as you generate the entries for the product matrix.



Source: openclipart.org

For example, there are three ways to score points in basketball: 1 point from a free throw, 2 points from a field goal, and 3 points from a long-range basket outside of a given curve. If you know how many of each that a team scored in a game, you can calculate the total points for that team. The table below shows the number of each type of score by two teams in a recent game.

	FREE THROWS	FIELD GOALS	3-POINTERS
SPURS	18	42	7
MAVERICKS	23	37	10

FREE THROWS	1 POINT
FIELD GOALS	2 POINTS
3-POINTERS	3 POINTS

From these tables, you can write two matrices.

$$\overset{\textbf{Matrix } S}{\begin{bmatrix} 18 & 42 & 7 \\ 23 & 37 & 10 \end{bmatrix}} \qquad \overset{\textbf{Matrix } P}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

You can use these matrices to calculate the total number of points scored by each team in matrix T. Multiply matrix S by matrix P.

$$S \times P = T$$

$$\begin{bmatrix} 18 & 42 & 7 \\ 23 & 37 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 18 \times 1 + 42 \times 2 + 7 \times 3 \\ 23 \times 1 + 37 \times 2 + 10 \times 3 \end{bmatrix} = \begin{bmatrix} 123 \\ 127 \end{bmatrix}$$

Notice that the product matrix T has the same number of rows as the first factor matrix and the same number of columns as the second factor matrix.

$$S_{2\times 3} \cdot P_{3\times 1} = SP_{2\times 1}$$

equal

In general, if you are multiplying matrix A with m rows and n columns by matrix B with n rows and p columns, the product matrix C will have m rows and p columns.

$$A_{m\times n} \cdot B_{n\times p} = C_{m\times p}$$

equal

You can also generalize multiplication of matrices. For multiplication of a 2×3 matrix by a 3×1 matrix, you can use variables for each entry.

$$\begin{bmatrix} a & b & c \\ d & f & g \end{bmatrix}\begin{bmatrix} h \\ j \\ k \end{bmatrix} = \begin{bmatrix} ah + bj + ck \\ dh + fj + gk \end{bmatrix}$$

For a $m \times n$ matrix multiplied by a $n \times p$ matrix, you can use specific paired matrix entries to generalize the product $m \times p$ matrix.

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}\begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,p} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,p} \end{bmatrix} = \begin{bmatrix} a_{1,1}b_{1,1} + a_{1,2}b_{2,1} + \cdots + a_{1,n}b_{n,1} & \cdots & a_{1,1}b_{1,p} + a_{1,2}b_{2,p} + \cdots + a_{1,n}b_{n,p} \\ \vdots & \ddots & \vdots \\ a_{m,1}b_{1,1} + a_{m,2}b_{2,1} + \cdots + a_{m,n}b_{n,1} & \cdots & a_{m,1}b_{1,p} + a_{m,2}b_{2,p} + \cdots + a_{m,n}b_{n,p} \end{bmatrix}$$

COMMUTATIVE PROPERTY WITH MATRICES

When you multiply real numbers, the order in which you multiply does not matter; $5.5 \times 7 = 7 \times 5.5$. However, with matrices, the order of multiplication does matter. If you multiply matrix A with 2 rows and 3 columns by matrix B with 3 rows and 1 column, then you have a product matrix with 2 rows and 1 column.

$$\begin{bmatrix} a & b & c \\ d & f & g \end{bmatrix}\begin{bmatrix} h \\ j \\ k \end{bmatrix} = \begin{bmatrix} ah + bj + ck \\ dh + fj + gk \end{bmatrix}$$

Multiplying matrix A by matrix B places matrix A on the left of matrix B and is called **left multiplication**. If you were to multiply matrix A on the right of matrix B, then it would be called **right multiplication**.

$$\begin{bmatrix} h \\ j \\ k \end{bmatrix}\begin{bmatrix} a & b & c \\ d & f & g \end{bmatrix} = ?$$

In this case the dimensions of the matrices do not match up and the multiplication is not possible. If you began with Row 1 and multiplied by Column 1, you can multiply the first pair of entries to get *ha* but there is no second entry in the first row of matrix *B* to pair with *d* in Column 1 of matrix *A*.

$$\underbrace{B_{3\times 1}\cdot A_{2\times 3}}_{\text{not equal}} = ?$$

MULTIPLICATION OF MATRICES

Two matrices can be multiplied together only if the number of columns in the first factor matrix is equal to the number of rows in the second factor matrix.

- Begin with Row 1 in the first matrix and Column 1 of the second matrix. Multiply each pair of entries, $a_{1,1} \times b_{1,1}$, $a_{1,2} \times b_{2,1}$, $a_{1,3} \times b_{3,1}$, and so on. Record the sum of these products in Row 1, Column 1 of the product matrix.
- Repeat the process for each pair of rows and columns.
- Multiplication of matrices is not commutative.
  - ✔ Left-multiplication means to take a matrix and multiply it by another matrix from the left.
  - ✔ Right-multiplication means to take a matrix and multiply it by another matrix from the right.

EXAMPLE 1

Determine whether or not the two matrices shown below can be multiplied. Justify your answer.

$$A = \begin{bmatrix} -1 & 0 \\ 3 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -3 \\ -8 & 1 \\ 4 & 5 \end{bmatrix}$$

STEP 1 **Determine the size of the matrices.**

Matrix *A* has two rows and two columns. $A_{2\times 2}$
Matrix *B* has three rows and two columns. $B_{3\times 2}$

STEP 2 **Use the size of the matrices to determine whether it is possible to multiply matrix *A* by matrix *B*.**

If it is possible to multiply AB, then the number of columns in matrix A must be equal to the number of rows in matrix B.

$$A_{2\times 2} \cdot B_{3\times 2} = ?$$

not equal

STEP 3 **Use the size of the matrices to determine whether it is possible to multiply matrix *B* by matrix *A*.**

If it is possible to multiply BA, then the number of columns in matrix B must be equal to the number of rows in matrix A.

$$B_{3\times 2} \cdot A_{2\times 2} = ?$$

equal

The sizes of the matrices make it impossible to multiply AB because the number of columns in matrix A is not equal to the number of rows in matrix B. The sizes of the matrices do make it possible to multiply BA since matrix B has the same number of columns as matrix A has rows.

YOU TRY IT! #1

Determine whether or not the two matrices shown below can be multiplied. Justify your answer.

$$C = \begin{bmatrix} 6 & -7 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 5 \end{bmatrix}$$

EXAMPLE 2

Multiply the two matrices to determine the product matrix EF.

$$E = \begin{bmatrix} 1 & -3 & 10 \end{bmatrix} \qquad F = \begin{bmatrix} 9 & -1 \\ -5 & -6 \\ 3 & 4 \end{bmatrix}$$

STEP 1 **Use the sizes of the factor matrices to determine the size of the product matrix so that you know how many entries will be in the product matrix.**

$$E_{1 \times 3} \cdot F_{3 \times 2} = EF_{1 \times 2}$$

equal

dimensions of product matrix

There will be two entries in the product matrix since the product matrix has one row and two columns.

STEP 2 **Multiply the entries in the row of matrix *E* by the corresponding entries in the first column of matrix *F*. Determine the sum of all these products to determine the entry in the first row and first column of the product matrix.**

$$EF = \begin{bmatrix} 1 & -3 & 10 \end{bmatrix} \cdot \begin{bmatrix} 9 & -1 \\ -5 & -6 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} (1)(9) + (-3)(-5) + (10)(3) & \boxed{\quad} \end{bmatrix} = \begin{bmatrix} 9 + 15 + 30 & \boxed{\quad} \end{bmatrix} = \begin{bmatrix} 54 & \boxed{\quad} \end{bmatrix}$$

STEP 3 **Multiply the entries in the row of matrix *E* by the corresponding entries in the second column of matrix *F*. Calculate the sum of all these products to determine the entry in the first row and second column of the product matrix.**

$$EF = \begin{bmatrix} 1 & -3 & 10 \end{bmatrix} \cdot \begin{bmatrix} 9 & -1 \\ -5 & -6 \\ 3 & 4 \end{bmatrix}$$

$$EF = \begin{bmatrix} 54 & (1)(-1) + (-3)(-6) + (10)(4) \end{bmatrix} = \begin{bmatrix} 54 & -1 + 18 + 40 \end{bmatrix} = \begin{bmatrix} 54 & 57 \end{bmatrix}$$

$$EF = \begin{bmatrix} 54 & 57 \end{bmatrix}$$

YOU TRY IT! #2

Multiply the two matrices to determine the product matrix *HG*.

$$G = \begin{bmatrix} 4 & 3 \\ 0 & -2 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & -7 \\ 8 & 12 \\ -3 & 0 \\ 0 & 4 \end{bmatrix}$$

EXAMPLE 3

Two friends both have family recipes for a traditional Spanish rice dish called paella. They compare the cost and protein content of their recipes.

	CHICKEN (POUNDS)	RICE (POUNDS)	SHELLFISH (POUNDS)
JAIME'S FAMILY RECIPE	2.5	0.75	2.25
MIA'S FAMILY RECIPE	1.5	3.5	0.5

	COST PER POUND (DOLLARS)	PROTEIN CONTENT PER POUND (GRAMS)
CHICKEN	$3.35	139 G
RICE	$0.70	12.3 G
SHELLFISH	$10.15	109 G

Whose paella recipe would you use if your goal were to eat a high-protein meal? Whose paella recipe would you use if your goal were to make a meal that cost less than $15? Justify your responses.

STEP 1 Write matrix R to represent the recipe ingredients and matrix C to represent the costs and protein contents per pound.

$$R = \begin{bmatrix} 2.5 & 0.75 & 2.25 \\ 1.5 & 3.5 & 0.5 \end{bmatrix} \qquad C = \begin{bmatrix} 3.35 & 139 \\ 0.7 & 12.3 \\ 10.15 & 109 \end{bmatrix}$$

STEP 2 Notice that the total cost and total protein content of each recipe can be found by multiplying the amount of each ingredient by its respective cost and protein content and adding the products. Therefore, you can multiply the two matrices to determine the total cost and total protein content of each paella recipe.

Multiply the entries in the first row of matrix R by the corresponding entries in the first column of matrix C. Determine the sum of all these products to determine the entry in the first row and first column of the product matrix.

$$RC = \begin{bmatrix} 2.5 & 0.75 & 2.25 \\ 1.5 & 3.5 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 3.35 & 139 \\ 0.7 & 12.3 \\ 10.15 & 109 \end{bmatrix}$$

$$\begin{bmatrix} (2.5)(3.35) + (0.75)(0.7) + (2.25)(10.15) & \square \\ \square & \square \end{bmatrix}$$

$$= \begin{bmatrix} 8.375 + 0.525 + 22.8375 & \square \\ \square & \square \end{bmatrix} = \begin{bmatrix} 31.7375 & \square \\ \square & \square \end{bmatrix}$$

Multiply the entries in the first row of matrix R by the corresponding entries in the second column of matrix C. Calculate the sum of all these products to determine the entry in the first row and second column of the product matrix.

$$RC = \begin{bmatrix} 2.5 & 0.75 & 2.25 \\ 1.5 & 3.5 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 3.35 & 139 \\ 0.7 & 12.3 \\ 10.15 & 109 \end{bmatrix}$$

$$\begin{bmatrix} 31.7375 & (2.5)(139) + (0.75)(12.3) + (2.25)(109) \\ \square & \square \end{bmatrix}$$

$$= \begin{bmatrix} 31.7375 & 347.5 + 9.225 + 245.25 \\ \square & \square \end{bmatrix} = \begin{bmatrix} 31.7375 & 601.975 \\ \square & \square \end{bmatrix}$$

Multiply the entries in the second row of matrix R by the corresponding entries in the first column of matrix C. Calculate the sum of all these products to determine the entry in the second row and first column of the product matrix.

$$RC = \begin{bmatrix} 2.5 & 0.75 & 2.25 \\ 1.5 & 3.5 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 3.35 & 139 \\ 0.7 & 12.3 \\ 10.15 & 109 \end{bmatrix}$$

$$\begin{bmatrix} 31.7375 & 601.975 \\ (1.5)(3.35) + (3.35)(0.7) + (0.5)(10.15) & \end{bmatrix}$$

$$= \begin{bmatrix} 31.7375 & 601.975 \\ 5.025 + 2.45 + 5.075 & \square \end{bmatrix} = \begin{bmatrix} 31.7375 & 601.975 \\ 12.55 & \square \end{bmatrix}$$

Multiply the entries in the second row of matrix *R* by the corresponding entries in the second column of matrix *C*. Calculate the sum of all these products to determine the entry in the second row and second column of the product matrix.

$$RC = \begin{bmatrix} 2.5 & 0.75 & 2.25 \\ 1.5 & 3.5 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 3.35 & 139 \\ 0.7 & 12.3 \\ 10.15 & 109 \end{bmatrix}$$

$$\begin{bmatrix} 31.7375 & 601.975 \\ 12.55 & (1.5)(139) + (3.5)(12.3) + (0.5)(109) \end{bmatrix}$$

$$RC = \begin{bmatrix} 31.7375 & 601.975 \\ 12.55 & 208.5 + 43.05 + 54.5 \end{bmatrix} = \begin{bmatrix} 31.7375 & 601.975 \\ 12.55 & 306.05 \end{bmatrix}$$

STEP 3 **Interpret the resulting product matrix.**

Jaime's family recipe for paella costs approximately $31.74 to make and contains about 602 grams of protein. Mia's family recipe for paella costs $12.55 to make and contains approximately 306 grams of protein. If my goal were to eat a high protein meal, I would use Jaime's family recipe for paella, but if my goal were to make a meal that cost less than $15 to make, I would use Mia's family recipe for paella.

YOU TRY IT! #3

Aimee and Christopher work at a local movie theater. They both sold tickets in the ticket booth on Tuesday. Which employee will have the higher receipt total at the end of their shifts? Create a matrix to organize the mathematical ideas, and justify your answer.

	NUMBER OF CHILD TICKETS SOLD	NUMBER OF STUDENT TICKETS SOLD	NUMBER OF ADULT TICKETS SOLD	NUMBER OF SENIOR TICKETS SOLD
AIMEE	60	48	144	30
CHRISTOPHER	102	27	108	60

	COST PER TICKET (DOLLARS)
CHILD	$7.25
STUDENT	$10.00
ADULT	$12.50
SENIOR	$8.00

PRACTICE/HOMEWORK

Use the scenario below to complete questions 1 – 4.

BUSINESS

Marcia owns a bakery where she bakes and sells bread, muffins, cakes, and pies. The table below represents the items she sells and the number of each item she sold for two weeks.

MARCIA'S BAKERY

	BREAD	MUFFINS	CAKES	PIES
WEEK 1	60	154	20	17
WEEK 2	68	147	23	15

The price she changes for each item is shown in the table below.

MARCIA'S PRICES

ITEM	PRICE
BREAD	$5.00
MUFFIN	$1.25
CAKE	$32.50
PIE	$14.00

1. Create matrix A to show the number of items Marcia sold in week 1 and week 2.

2. Create matrix B to show the price for each item sold in the bakery.

3. How should the matrices be multiplied in order to generate a matrix representing the total revenue for each week? Choose one and justify your answer.
A. $[A] \cdot [B]$ **B.** $[B] \cdot [A]$

4. Multiply the matrices to generate a matrix representing the total revenue for each week.

Use the scenario below to complete questions 5 – 8.

SPORTS

There are several ways to score points in professional football: touchdown, field goal, safety, extra point after a touchdown, and conversion after a touchdown. The table below shows the number of each type of score by two teams.

FOOTBALL TEAMS

	TOUCHDOWN	FIELD GOAL	SAFETY	EXTRA POINT	CONVERSION
TEAM C	3	2	0	2	1
TEAM B	4	1	1	3	0

The table below shows the points awarded for each type of score.

POINTS

TYPE	POINTS
TOUCHDOWN	6
FIELD GOAL	3
SAFETY	2
EXTRA POINT	1
CONVERSION	2

5. Create matrix C to show the number of each type of score made by both teams.

6. Create matrix B to show the points awarded for each type of score.

7. How should the matrices be multiplied in order to generate a matrix representing the total points for each team? Choose one and justify your answer.
A. $[B] \cdot [C]$ **B.** $[C] \cdot [B]$

8. Multiply the matrices to generate a matrix representing the total points for each team.

For questions 9 – 14, write 'yes' if the matrices can be multiplied in the order listed or 'no' if they cannot. If 'yes', indicate the size of the product matrix.

9. $X_{4x3} \cdot Y_{3x3}$ **10.** $R_{5x1} \cdot T_{5x7}$ **11.** $K_{6x2} \cdot L_{6x4}$ **12.** $B_{7x4} \cdot D_{4x7}$

13. $A_{1x3} \cdot C_{1x2}$ **14.** $F_{5x6} \cdot G_{6x8}$

Use the matrices below for questions 15 – 20.

$$A = \begin{bmatrix} -3 & 1 \\ 7 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -6 & 10 \\ -9 & 0 & -5 \\ 5 & -1 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 0 & -4 \\ 6 & -5 \\ 8 & -10 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 9 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 3 & -2 & 5 \\ 6 & -4 & 8 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 6 \\ -4 \end{bmatrix} \quad G = \begin{bmatrix} 7 & -7 \\ 2 & 0 \\ -3 & 2 \\ 5 & 6 \end{bmatrix} \quad H = \begin{bmatrix} 5 & -6 & 2 \end{bmatrix}$$

Perform each multiplication, if possible, and record the product matrix. If it is not possible to multiply, write 'no solution'.

15. BC **16.** EG **17.** AH **18.** DC

19. AC **20.** FH

Solving Systems of Two Linear Equations

6.5

FOCUSING QUESTION How can I use matrices to represent and solve a system of linear equations?

LEARNING OUTCOMES

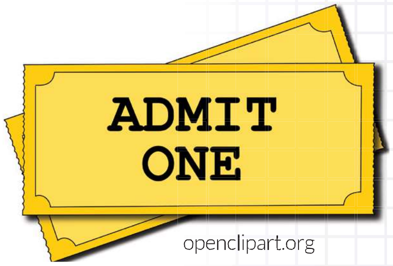
- I can represent a problem with systems of linear equations using matrices.
- I can solve a problem with systems of linear equations using matrices.
- I can select tools, including paper and pencil and technology, as appropriate to solve problems.

ENGAGE

The Texas Cowboy Hall of Fame in Fort Worth, Texas, charges \$5 for adult admission and \$3 for children ages 5-12. A group of 12 people visits the museum and pays \$56 for admission. Write a system of equations you could use to determine *c*, the number of children and *a*, the number of adults in the group.

EXPLORE

The O'Neal High School drama club raised money for Project Prom by producing a play that was open to the community. On Friday night, they collected \$1,344 from 64 student tickets and 96 adult tickets. On Saturday night, they collected \$1,464 from 80 student tickets and 96 adult tickets. What is the cost of one student ticket and the cost of one adult ticket?



openclipart.org

1. Let x represent the cost of one student ticket and y represent the cost of one adult ticket. Write a system of equations that you could use to represent this problem.

2. Use the coefficients of x and y to write matrix A where each row represents one equation and each column represents one of the variables x or y.

3. Use the constants from each equation to write matrix B where each row represents one equation.

4. To represent a system of equations with matrices, use the matrix $\begin{bmatrix} x \\ y \end{bmatrix}$ to represent the variables. Write a matrix equation that multiplies the coefficient matrix A by the variable matrix to equal the constant matrix B.

5. If you had an equation $Ax = B$, where A and B were real numbers, how would you solve for x?

6. How are the operations of division and multiplication related?

7. To solve the matrix equation what should you multiply by in order to solve for x? Remember that matrix multiplication is not commutative, so be sure to specify whether the multiplication should be left-multiplication or right-multiplication.

8. Write a matrix equation relating A, A^{-1}, $\begin{bmatrix} x \\ y \end{bmatrix}$, and B.

One way to indicate the **inverse** of a number, variable, function, or matrix is to use an exponent of −1. For example, the inverse of $f(x)$ is $f^{-1}(x)$ and the multiplicative inverse of 3 is $3^{-1} = \frac{1}{3}$.

The inverse of a 2×2 matrix can be calculated using the formula shown.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

9. Use the inverse formula to calculate the matrix A^{-1}.

10. Use the matrix A^{-1} and the matrix B to solve the matrix equation from the previous question for $\begin{bmatrix} x \\ y \end{bmatrix}$. Simplify the product completely.

11. Use graphing technology such as a graphing calculator or app to enter the data from matrix A and matrix B. Remember that matrices are defined by the number of rows by the number of columns.

12. Use matrix operations to left-multiply the inverse of matrix A by matrix B, $[A]^{-1} \times [B]$. How does this matrix compare to matrix $\begin{bmatrix} x \\ y \end{bmatrix}$, the one you calculated in a previous question?

13. The matrix for $\begin{bmatrix} x \\ y \end{bmatrix}$ identifies the values of x and y in the system of equations. Write the solution to the system and explain what it means in the context of the original problem.

REFLECT

- **The matrix equation $[A]\begin{bmatrix} x \\ y \end{bmatrix} = [B]$ represents the system of linear equations relating x and y. What operation do you need to do to this equation to solve for $\begin{bmatrix} x \\ y \end{bmatrix}$? Why?**
- **Why do you left-multiply by the inverse of matrix A instead of right-multiplying?**

EXPLAIN

Certain types of real-world situations can be represented using systems of equations. Systems of equations are used for situations with multiple unknowns when you are given certain pieces of information about how those unknowns are related.

Watch Explain and You Try It Videos

or click here

There are also several ways to solve systems of linear equations with two variables. For example, consider the problem shown.

> The perimeter of a rectangle is 8 feet. The length of the rectangle is 5 feet less than twice the width. What are the length and width of the rectangle?

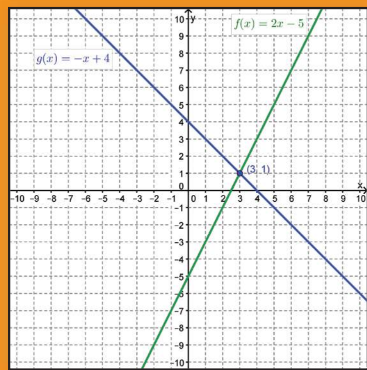
In previous courses, you learned that systems of linear equations can be solved with graphs, tables, and algebraic methods including substitution and elimination.

ELPS CONNECTION

Study the graphic organizer shown. With a partner, use the visual graphic organizer along with the contextual support of the perimeter problem to study the different methods that could be used to solve the perimeter problem. How does the visual and contextual support enhance and confirm your understanding of systems of equations?

GRAPHING

Graph both equations and look for the point of intersection.



TABLES

Make a table of values for both equations. Look for the *x*-value that generates the same *y*-value for both equations.

x	$Y1$	$Y2$
2	-1	2
3	1	1
4	3	0

SOLVING SYSTEMS OF LINEAR EQUATIONS

SUBSTITUTION

Solve one equation for one variable in terms of the other. Substitute this expression into the second equation.

$$\boxed{\begin{cases} y = 2x - 5 \\ 2x + 2y = 8 \end{cases}}$$

$$\begin{aligned} 2x + 2(2x - 5) &= 8 \\ 2x + 4x - 10 &= 8 \\ 6x - 10 &= 8 \\ 6x &= 18 \\ x &= 3 \\ y = 2(3) - 5 &= 1 \end{aligned}$$

ELIMINATION OR ADDITION

Place both equations in standard form. Multiply one equation by an integer that will cause one variable to sum to 0 when added to the second equation.

$$\boxed{\begin{cases} y = 2x - 5 \\ 2x + 2y = 8 \end{cases}}$$

$$\begin{array}{rr} 2x - y = 5 \rightarrow & 2x - y = 5 \\ -(2x + 2y = 8) & -2x - 2y = -8 \\ & -3y = -3 \\ 2x + 2(1) = 8 & y = 1 \\ 2x + 2 = 8 & \\ 2x = 6 & \\ x = 3 & \end{array}$$

There are other ways to solve systems of linear equations. One way is to use matrices.

DETERMINING THE INVERSE OF A MATRIX

When representing and solving systems of linear equations using matrices, you must use the inverse of a matrix. Technology can calculate the inverse of a matrix, but you can also perform these calculations by hand.

For a 2×2 matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the product of matrix A and the inverse of matrix A, which is written as A^{-1}, must be equal to the identity matrix.

$$A^{-1} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solving this matrix equation for A^{-1} generates a formula for determining A^{-1} from a given matrix A.

The product of a number and its multiplicative inverse is 1. For example, the multiplicative inverse of $\frac{2}{5}$ is $\frac{5}{2}$ because $\frac{2}{5} \times \frac{5}{2} = \frac{10}{10} = 1$. Likewise, the product of a matrix and its inverse is the identity matrix consisting only of 1's and 0's. For a 2×2 matrix, the **identity matrix** is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

REPRESENTING A SYSTEM OF LINEAR EQUATIONS USING MATRICES

For a system of two linear equations with two unknowns, you can use a matrix equation with 2×2 matrices to represent the system. First, make sure that both linear equations are in standard form, $Ax + By = C$. Then, place the coefficients of the unknowns into a 2×2 coefficient matrix, the unknowns into a 2×1 variable matrix, and the constants (C when the equation is in standard form) into a 2×1 constant matrix.

Let's look back at the perimeter problem. If x represents the width of the rectangle and y represents the length of the rectangle, then you can write the system of two linear equations shown.

$$\begin{cases} y = 2x - 5 \\ x + y = 4 \end{cases} \longrightarrow \begin{cases} 2x - y = 5 \\ x + y = 4 \end{cases} \longrightarrow \underbrace{\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}}_{\text{coefficient matrix}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{variable matrix}} = \underbrace{\begin{bmatrix} 5 \\ 4 \end{bmatrix}}_{\text{constant matrix}}$$

SOLVING A SYSTEM OF LINEAR EQUATIONS USING MATRICES

Once you have written your matrix equation, you can use the inverse of the coefficient matrix to solve for the variable matrix. Remember that matrix multiplication is not commutative. When you use matrix multiplication to solve a matrix equation, there is a variable matrix. You will need to place the inverse matrix on the left side of the equation making left-multiplication necessary on both sides of the equation.

$$[A]\begin{bmatrix} x \\ y \end{bmatrix} = [B] \longrightarrow [A]^{-1}[A]\begin{bmatrix} x \\ y \end{bmatrix} = [A]^{-1}[B]$$

For example, if you have the matrix equation shown, you can calculate the inverse of the coefficient matrix and begin multiplication. In this equation, the original perimeter equation from the perimeter problem, $2x + 2y = 8$ has been simplified to $x + y = 4$.

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

The inverse of $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ can be calculated using the inverse formula.

$$[A]^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$[A]^{-1} = \frac{1}{2(1) - (-1)(1)}\begin{bmatrix} 1 & -(-1) \\ -(1) & 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Use $[A]^{-1}$ to solve the original matrix equation for the variable matrix, $\begin{bmatrix} x \\ y \end{bmatrix}$.

$$[A]^{-1}[A]\begin{bmatrix} x \\ y \end{bmatrix} = [A]^{-1}[B]$$

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3}(5) + \frac{1}{3}(4) \\ -\frac{1}{3}(5) + \frac{2}{3}(4) \end{bmatrix}$$

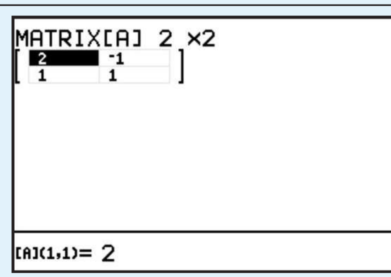
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3}(5) + \frac{1}{3}(4) \\ -\frac{1}{3}(5) + \frac{2}{3}(4) \end{bmatrix}$$

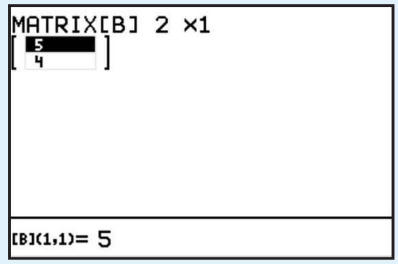
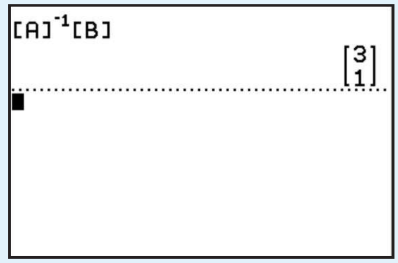
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{9}{3} \\ \frac{3}{3} \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

The solution to the given system is (3, 1). In the context of the problem, the rectangle has a width of 3 feet and a length of 1 foot.

USING TECHNOLOGY TO SOLVE SYSTEMS OF TWO LINEAR EQUATIONS WITH MATRICES

Graphing technology, such as a graphing calculator or app, can be extremely beneficial when solving systems of equations with matrices.

Write the system of linear equations as a matrix equation. Make sure that both linear equations are in standard form.	$\begin{cases} y = 2x - 5 \\ x + y = 4 \end{cases} \longrightarrow \begin{cases} 2x - y = 5 \\ x + y = 4 \end{cases}$ $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$
Enter the coefficient matrix into one matrix of your graphing technology.	

Enter the constant matrix into a second matrix of your graphing technology.	
Use matrix operations to calculate $[A]^{-1}[B]$.	

SYSTEMS OF TWO LINEAR EQUATIONS WITH MATRICES

Matrices can be used to represent and solve systems of two linear equations.

- Make sure that both linear equations are in standard form, $Ax + By = C$, where A, B, and C are integers, $A \neq 0$, and $B \neq 0$.
- Use the matrix equation $[A]\begin{bmatrix} x \\ y \end{bmatrix} = [B]$ where $[A]$ represents the coefficient matrix and $[B]$ represents the constant matrix.
- Determine the inverse of the coefficient matrix, $[A]^{-1}$.
- Left-multiply both sides of the matrix equation by $[A]^{-1}$. In the left member of the equation, $[A]^{-1}[A] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is the identity matrix.
- Technology can be used to enter and calculate the values of the variable matrix, $\begin{bmatrix} x \\ y \end{bmatrix} = [A]^{-1}[B]$.

EXAMPLE 1

Two angles are supplementary. The difference in their measures is thirty degrees. What are the measures of the two angles? Write a system of linear equations and its corresponding matrix equation to represent the situation. Solve the system using matrices.

STEP 1 Define the variables to represent the unknowns and use them to write a system of linear equations that represents the situation.

Since the problem asks you to determine the measures of the angles, the variables will represent the measures in degrees of the two angles. Let x represent the measure in degrees of the larger angle, and let y represent the measure in degrees of the smaller angle.

"The two angles are supplementary" → $x + y = 180$.
"The difference in their measures is thirty degrees." → $x - y = 30$.

Therefore, a system of linear equations that represents the situation is

$$\begin{cases} x + y = 180 \\ x - y = 30 \end{cases}$$

STEP 2 **Write a matrix equation that corresponds to the system of linear equations you wrote in Step 1. If necessary, rewrite any linear equations that are not already in standard form before writing the matrix equation.**

Both linear equations in the system you wrote in Step 1 are in standard form, so it is not necessary to rewrite either equation.

An understood coefficient of one is implied for both x and y.

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 180 \\ 30 \end{bmatrix}$$

STEP 3 **Solve the system using matrices by finding the inverse of the coefficient matrix and left-multiplying both sides of the matrix equation by it.**

Recall that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Since $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $A^{-1} = \frac{1}{(1)(-1)-(1)(1)}\begin{bmatrix} -1 & -(1) \\ -(1) & 1 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

Use the inverse of the coefficient matrix to solve the system.

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 180 \\ 30 \end{bmatrix}$$

$$\begin{bmatrix} \left(\frac{1}{2}\right)(1) + \left(\frac{1}{2}\right)(1) & \left(\frac{1}{2}\right)(1) + \left(\frac{1}{2}\right)(-1) \\ \left(\frac{1}{2}\right)(1) + \left(-\frac{1}{2}\right)(1) & \left(\frac{1}{2}\right)(1) + \left(-\frac{1}{2}\right)(-1) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}\right)(180) + \left(\frac{1}{2}\right)(30) \\ \left(\frac{1}{2}\right)(180) + \left(-\frac{1}{2}\right)(30) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 90+15 \\ 90-15 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 105 \\ 75 \end{bmatrix}$$

$\begin{cases} x+y=180 \\ x-y=30 \end{cases}$ *is the system of linear equations that represents this situation if x is the measure in degrees of the larger angle and y is the measure in degrees of the smaller angle.*

$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 180 \\ 30 \end{bmatrix}$ *is the matrix equation that represents the system of equations*

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 105 \\ 75 \end{bmatrix}$ *is the solution for the system determined using matrices.*

The measure of the larger angle is 105 degrees and the measure of the smaller angle is 75 degrees.

YOU TRY IT! #1

Two angles are complementary. Three times the measure of the larger angle is ten degrees less than four times the measure of the smaller angle. Write a system of linear equations and its corresponding matrix equation to represent the system. Solve the system using matrices.

EXAMPLE 2

A high school's football team won its first game with the help of its kicker. He scored a total of nine points in field goals and extra points after touchdowns to help his team win by a narrow two-point margin. His tough week of practice paid off. He attempted to score five times in the game and kicked the football through the uprights each time. How many extra points and how many field goals did the high school football kicker kick in the game that he helped his team win? Write a system of linear equations and its corresponding matrix equation to represent the system. Solve the system using matrices.

STEP 1 Define variables to represent the unknowns and use them write a system of linear equations to represent the system.

Let t represent the number of extra points and f represent the number of field goals that the kicker kicked to help his team win the game.

"He scored a total of nine points…". Since extra points after touchdowns are worth a single point and field goals are worth three points, then $t + 3f = 9$.
"He attempted to score five times…and kicked the football through the uprights each time." Therefore, $t + f = 5$.

A system of linear equations that represents the situation is

$$\begin{cases} t + 3f = 9 \\ t + f = 5 \end{cases}$$

STEP 2 **Write a matrix equation that corresponds to the system of linear equations you wrote in Step 1. If necessary, rewrite any linear equations that are not already in standard form before writing the matrix equation.**

Both linear equations in the system you wrote in Step 1 are in standard form, so it is not necessary to rewrite either equation.

$$\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} t \\ f \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$

STEP 3 **Solve the system using matrices by finding the inverse of the coefficient matrix and left-multiplying both sides of the matrix equation by it.**

Recall that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Since $A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$, $A^{-1} = \frac{1}{(1)(1)-(3)(1)}\begin{bmatrix} 1 & -(3) \\ -(1) & 1 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 1 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

Use the inverse of the coefficient matrix to solve the system.

$$\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} t \\ f \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} \left(-\frac{1}{2}\right)(1) + \left(\frac{3}{2}\right)(1) & \left(-\frac{1}{2}\right)(3) + \left(\frac{3}{2}\right)(1) \\ \left(\frac{1}{2}\right)(1) + \left(-\frac{1}{2}\right)(1) & \left(\frac{1}{2}\right)(3) + \left(-\frac{1}{2}\right)(1) \end{bmatrix}\begin{bmatrix} t \\ f \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2}\right)(9) + \left(\frac{3}{2}\right)(5) \\ \left(\frac{1}{2}\right)(9) + \left(-\frac{1}{2}\right)(5) \end{bmatrix}$$

$$\begin{bmatrix} \left(-\frac{1}{2}\right)+\left(\frac{3}{2}\right) & \left(-\frac{3}{2}\right)+\left(\frac{3}{2}\right) \\ \left(\frac{1}{2}\right)+\left(-\frac{1}{2}\right) & \left(\frac{3}{2}\right)+\left(-\frac{1}{2}\right) \end{bmatrix}\begin{bmatrix} t \\ f \end{bmatrix}=\begin{bmatrix} -\frac{9}{2}+\frac{15}{2} \\ \frac{9}{2}-\frac{5}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} t \\ f \end{bmatrix}=\begin{bmatrix} \frac{6}{2} \\ \frac{4}{2} \end{bmatrix}$$

$$\begin{bmatrix} t \\ f \end{bmatrix}=\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$\begin{cases} t+3f=9 \\ t+f=5 \end{cases}$ ***is the system of linear equations that represents this situation if t is the number of extra points kicked after touchdowns and f is the number of field goals kicked.***

$\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} t \\ f \end{bmatrix}=\begin{bmatrix} 9 \\ 5 \end{bmatrix}$ ***is the matrix equation that represents the system of equations.***

$\begin{bmatrix} t \\ f \end{bmatrix}=\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ ***is the solution for the system determined using matrices.***

The high school football team's kicker made three successful extra point kicks after touchdowns in addition to two field goals.

YOU TRY IT! #2

A teacher prepares a side dish of peas and carrots to bring to a faculty potluck luncheon. The recipe calls for six cups of vegetables. The teacher spends \$7.89 at a grocery store to purchase fresh carrots for \$1.28 per cup and peas for \$1.34 per cup. How many cups of peas and carrots are in the teacher's side dish? Write a system of linear equations and its corresponding matrix equation to represent the system. Solve the system using matrices.

EXAMPLE 3

Charlotte and Kaylee sing in their high school's choir. For this year's fundraiser, the choir director has chosen to sell scented bars of soap in addition to the usual plastic cups with the high school's logo on them. Charlotte sold 126 bars of soap and 64 plastic cups, raising a total of $1,350. Kaylee raised $1,340 by selling 43 bars of soap and 100 plastic cups. How much did the choir charge for bars of soap and plastic cups? Select pencil and paper or technology as appropriate to represent and solve this problem using matrices with a system of linear equations.

STEP 1 **Select paper and pencil or technology to define variables to represent the unknowns and use them write a system of linear equations to represent the system.**

You can choose to use paper and pencil to define variables and write a system of equations. Let x represent the price of a bar of soap and y represent the price of a plastic cup the choir members sold for this year's fundraiser.

The equation that represents Charlotte's sales is $126x + 64y = 1350$.
The equation that represents Kaylee's sales is $43x + 100y = 1340$.

A system of linear equations that represents the situation is

$$\begin{cases} 126x + 64y = 1350 \\ 43x + 100y = 1340 \end{cases}$$

STEP 2 **Write a matrix equation that corresponds to the system of linear equations you wrote in Step 1. If necessary, rewrite any linear equations that are not already in standard form before writing the matrix equation.**

Both linear equations in the system you wrote in Step 1 are in standard form, so it is not necessary to rewrite either equation.

$$\begin{bmatrix} 126 & 64 \\ 43 & 100 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1350 \\ 1340 \end{bmatrix}$$

STEP 3 **Solve the system using matrices.**

The values in the coefficient matrix and constant matrix are quite large. While you could use pencil and paper to find the inverse matrix and left-multiply it to both sides of the matrix equation, it would take a significant amount of time to do so. You can choose to use technology to find the inverse matrix and also to multiply it by the constant matrix.

The example below shows the problem worked out using pencil and paper. Refer to the part of this section entitled, **USING TECHNOLOGY TO SOLVE SYSTEMS OF TWO LINEAR EQUATIONS WITH MATRICES** to recall how technology may be used instead.

Recall that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Let $A = \begin{bmatrix} 126 & 64 \\ 43 & 100 \end{bmatrix}$.

$$A^{-1} = \frac{1}{(126)(100)-(64)(43)}\begin{bmatrix} 100 & -(64) \\ -(43) & 126 \end{bmatrix} = \frac{1}{9848}\begin{bmatrix} 100 & -64 \\ -43 & 126 \end{bmatrix}$$

Use the inverse of the coefficient matrix to solve the system.

$$\frac{1}{9848}\begin{bmatrix} 100 & -64 \\ -43 & 126 \end{bmatrix} \cdot \begin{bmatrix} 126 & 64 \\ 43 & 100 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{9848}\begin{bmatrix} 100 & -64 \\ -43 & 126 \end{bmatrix} \cdot \begin{bmatrix} 1350 \\ 1340 \end{bmatrix}$$

$$\frac{1}{9848}\begin{bmatrix} (100)(126) + (-64)(43) & (100)(64) + (-64)(100) \\ (-43)(126) + (126)(43) & (-43)(64) + (126)(100) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{9848}\begin{bmatrix} (100)(1350) + (-64)(1340) \\ (-43)(1350) + (126)(1340) \end{bmatrix}$$

$$\frac{1}{9848}\begin{bmatrix} 12600 + (-2752) & 6400 + (-6400) \\ (-5418) + 5418 & (-2752) + 12600 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{9848}\begin{bmatrix} 135000 + (-85760) \\ (-58050) + 168840 \end{bmatrix}$$

$$\frac{1}{9848}\begin{bmatrix} 9848 & 0 \\ 0 & 9848 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{9848}\begin{bmatrix} 49240 \\ 110790 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ \frac{45}{4} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 11.25 \end{bmatrix}$$

Charlotte and Kaylee's high school choir charged \$5.00 for each scented bar of soap and \$11.25 for each plastic cup with the high school's logo printed on it for this year's fundraiser.

YOU TRY IT! #3

After a year, a teacher's investments in an individual retirement account (IRA) that pays 4% interest and a savings account that pays 1.5% interest have earned \$125. The teacher's invested a total of \$5,000. Select pencil and paper or technology as appropriate to represent and solve this problem using matrices with a system of linear equations.

PRACTICE/HOMEWORK

For questions 1 – 8, write a matrix equation to represent the system of linear equations and solve the system using matrices.

1. $\begin{cases} x + y = 9 \\ x - y = 3 \end{cases}$

2. $\begin{cases} 2x + 3y = 12 \\ 2x - y = 4 \end{cases}$

3. $\begin{cases} 3x + 5y = -4 \\ 4x + 2y = 18 \end{cases}$

4. $\begin{cases} 5x + 2y = -5 \\ 7x + 3y = -6 \end{cases}$

5. $\begin{cases} x - 2y = 7 \\ 2x - 7y = 11 \end{cases}$

6. $\begin{cases} 1.5x - 0.2y = 8.3 \\ 2.5x + 1.4y = 6.9 \end{cases}$

7. $\begin{cases} \frac{1}{2}x - \frac{1}{4}y = 3 \\ \frac{3}{4}x + \frac{1}{2}y = 1 \end{cases}$

8. $\begin{cases} 3x - 7y = -5 \\ x + 3y = 1 \end{cases}$

Use the following scenario for questions 9 - 11.

FINANCE

Tickets for a soccer game are \$6 for adults and \$4 for students. At the last game there were 220 tickets sold and \$1064 collected.

9. Write a system of linear equations to represent the situation.

10. Write a matrix equation that corresponds to the system of linear equations you wrote in question 9.

11. Solve the system using matrices to determine how many adult tickets and student tickets were sold.

Use the following scenario for questions 12 - 14.

FINANCE

Allison has \$5.10 in quarters and dimes in her piggy bank. She has 27 coins in all.

12. Write a system of linear equations to represent the situation.

13. Write a matrix equation that corresponds to the system of linear equations you wrote in question 12.

14. Solve the system using matrices to determine how many quarters and dimes Allison has in her piggy bank.

Use the following scenario for questions 15 - 17.

CRITICAL THINKING

A test has 28 questions that total 100 points. The test contains multiple choice questions that are worth 3 points each and short answer questions that are worth 5 points each.

15. Write a system of linear equations to represent the situation.

16. Write a matrix equation that corresponds to the system of linear equations you wrote in question 15.

17. Solve the system using matrices to determine how many multiple choice questions and how many short answer questions are on the test.

Use the following scenario for questions 18 - 20.

CRITICAL THINKING

Ryan is playing a number game with his little brother Andrew. Ryan said he was thinking of two numbers and wanted Andrew to figure out the two numbers. Ryan said that two times the smaller number plus three times the larger number is forty-five. Also, three times the smaller number plus two times the larger number is forty.

18. Write a system of linear equations to represent the situation.

19. Write a matrix equation that corresponds to the system of linear equations you wrote in question 18.

20. Solve the system using matrices to determine what the two numbers are that Ryan is thinking about.

6.6 Solving Systems of Three Linear Equations

FOCUSING QUESTION How can I use matrices and technology to represent and solve a system of three linear equations?

LEARNING OUTCOMES

- I can represent a problem with a system of three linear equations using matrices and technology.
- I can use matrices and technology to solve a problem involving a system of three linear equations.
- I can select tools, including paper and pencil and technology, as appropriate to solve

ENGAGE

Lashondra owns a Christmas tree farm. She wants to plant both Douglas fir trees and Scotch pine trees to sell next Christmas. Douglas fir trees cost \$250 per acre to plant and Scotch pine trees cost \$175 per acre to plant. Lashondra will plant 50 acres of trees and will spend \$11,000. How many acres of each tree will Lashondra plant? Define your variables, write a system of equations, and represent the system using a matrix equation.

EXPLORE

Image source: Pixabay

Gulf Stream Lumber has a plant with three sawmills, A, B, and C. If all three sawmills run all day, then the three sawmills produce 5,700 board-feet of lumber. If sawmill A runs for two days and sawmill B runs for one day, they produce 4,900 board-feet of lumber. If only sawmills B and C run all day, then the two sawmills together produce 4,200 board-feet of lumber.

1. Write a system of equations that you could use to represent this problem. Let the variables a, b, and c represent the amount of lumber produced in one day by their respective sawmills.

2. Use the coefficients of a, b, and c to write matrix A where each row represents one equation and each column represents one of the variables a, b, and c. If an equation does not contain all three variables, be sure to use a term with 0 as a coefficient as a placeholder.

3. Use the constants from each equation to write matrix B where each row represents one equation.

4. Use the matrix $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to represent the variables and write a matrix equation relating matrix A, matrix B, and the variable matrix.

The inverse of a 3 × 3 matrix can be calculated using the formula shown.

If $A = \begin{bmatrix} a & b & c \\ d & f & g \\ h & j & k \end{bmatrix}$, then $A^{-1} = \dfrac{1}{a(fk-gj)-b(dk-gh)-c(dj-fh)} \begin{bmatrix} fk-gj & cj-bk & bg-cf \\ gh-dk & ak-ch & cd-ag \\ dj-fh & bh-aj & af-bd \end{bmatrix}$.

5. Use the inverse formula with paper and pencil, graphing technology (e.g., calculator or app), or an online matrix inverse calculator to calculate the matrix A^{-1}.

6. Use the matrix A^{-1} and the matrix equation from a previous question to solve for $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

7. Write the solution to the system and explain what it means in the context of the original problem.

Repetitive computations such as those involved with calculating the inverse of a 3 × 3 matrix are easily programmed into a computer. Coding algorithms such as calculating the inverse of a matrix have many applications for computer programming.

Use the situation below to answer questions 8 – 10.

A candy company packages chocolate gift boxes using white chocolate, milk chocolate, and dark chocolate.

- A box of 1 package of white chocolate, 3 packages of milk chocolate, and 3 packages of dark chocolate weighs 14 ounces.
- A box of 1 package of white chocolate, 3 packages of milk chocolate, and 4 packages of dark chocolate weighs 17 ounces.
- A box of 1 package of white chocolate, 4 packages of milk chocolate, and 3 packages of dark chocolate weighs 15 ounces.

8. Let x represent the weight of one package of white chocolate, y represent the weight of one package of milk chocolate, and z represent the weight of one package of dark chocolate. Write a system of three linear equations that you can use to represent this problem.

9. Use technology to represent the system of three linear equations.

10. Use technology to solve the system of three linear equations. Interpret the solution within the context of the problem.

REFLECT

- **When solving a system of three linear equations using matrices, is it easier to use paper and pencil or technology to solve the problem? Explain your reasoning.**

- **How is the idea of left-multiplying by the inverse matrix to solve a matrix equation related to multiplying by the inverse to solve an equation with real numbers (e.g., to solve $2x = 7$, multiply by the inverse of 2)?**

EXPLAIN

Situations with three unknowns require particular pieces of information in order to solve for the values of those unknowns. To write a system of equations for the situation, you need to have as many equations as you do unknowns in order to solve the system. If the equations are all linear equations, then you can use matrices to solve the system of three linear equations.

Watch Explain and You Try It Videos

or click here

ELPS ACTIVITY

Read the material in this section with a partner. Use support from your peer by asking each other the following questions.

- Could you please summarize what this paragraph is saying?
- What word that you know is similar to this word?
- Where have you seen something like this before?

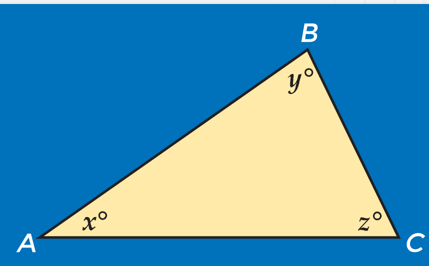
REPRESENTING A SYSTEM OF THREE LINEAR EQUATIONS USING MATRICES

For a system of three linear equations with three unknowns, you can use a matrix equation with 3×3 matrices to represent the system. This process is similar to what you did with systems of two linear equations with two unknowns.

- Make sure that all linear equations are in standard form, $Ax + By + Cz = D$. If one variable is missing from the equation, be sure to use a term with 0 as the coefficient as a placeholder.

- Place the coefficients of the unknowns into a 3 × 3 coefficient matrix.
- Place the unknowns into a 3 × 1 variable matrix.
- Place the constants (D when the equation is in standard form) into a 3 × 1 constant matrix.

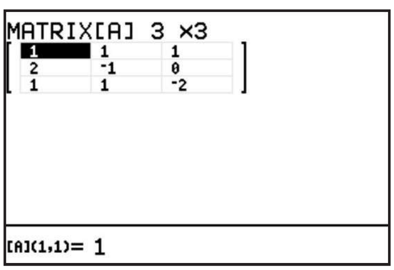
Consider the triangle problem shown. If x, y, and z each represent the measure of one interior angle of ΔABC, then you can write the system of three linear equations shown.

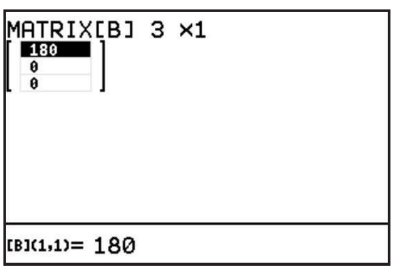


In ΔABC, the sum of the measures of the interior angles is 180°. Also, $m\angle B$ is twice $m\angle A$ and the sum of $m\angle A$ and $m\angle B$ is twice $m\angle C$. What are the measures of each interior angle of ΔABC?

$$\begin{cases} x + y + z = 180 \\ y = 2x \\ x + y = 2z \end{cases} \longrightarrow \begin{cases} x + y + z = 180 \\ 2x - y + 0z = 0 \\ x + y - 2z = 0 \end{cases} \longrightarrow \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix}}_{\text{coefficient matrix}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\text{variable matrix}} = \underbrace{\begin{bmatrix} 180 \\ 0 \\ 0 \end{bmatrix}}_{\text{constant matrix}}$$

Once you have the system of three linear equations represented in a matrix, you can use technology to represent the system. In a graphing calculator, you enter the numeric values of each matrix entry into a matrix app.





DETERMINING THE INVERSE OF A 3 × 3 MATRIX

As you have seen with systems of two linear equations, in order to solve the matrix equation for the variable matrix, you need to left-multiply by the inverse of the coefficient matrix. In this case, the coefficient matrix is a 3 × 3 matrix.

For a 3 × 3 matrix, $A = \begin{bmatrix} a & b & c \\ d & f & g \\ h & j & k \end{bmatrix}$, the product of the inverse of matrix A, which is written as A^{-1}, and matrix A must be equal to the identity matrix.

$$A^{-1} \times \begin{bmatrix} a & b & c \\ d & f & g \\ h & j & k \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving this matrix equation for A^{-1} generates a formula for determining A^{-1} from a given matrix A.

If $A = \begin{bmatrix} a & b & c \\ d & f & g \\ h & j & k \end{bmatrix}$, then $A^{-1} = \dfrac{1}{a(fk-gj)-b(dk-gh)-c(dj-fh)} \begin{bmatrix} fk-gj & cj-bk & bg-cf \\ gh-dk & ak-ch & cd-ag \\ dj-fh & bh-aj & af-bd \end{bmatrix}$.

You can certainly calculate the inverse of a 3 × 3 matrix using paper and pencil and a calculator by hand. Or, you can use graphing technology, an app, or an online matrix inverse calculator to calculate the inverse of the coefficient matrix.

SOLVING A SYSTEM OF THREE LINEAR EQUATIONS USING MATRICES

Once you have written your matrix equation, you can use the inverse of the coefficient matrix to solve for the variable matrix. Remember that matrix multiplication is not commutative. So if you want the product of $[A]^{-1}$ and $[A]$ to equal the identity matrix, then you will need to left-multiply $[A]^{-1}$ by $[A]$.

$$[A]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [B] \longrightarrow [A]^{-1}[A]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [A]^{-1}[B]$$

With the triangle problem, you represented the situation with a system of three linear equations and then used the equations to create a matrix equation. You also used technology to represent the system of equations through matrices. Now, you can calculate the inverse of the coefficient matrix and begin multiplication.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 180 \\ 0 \\ 0 \end{bmatrix}$$

The inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix}$ can be calculated using either the inverse formula or technology.

$$[A]^{-1} = \begin{bmatrix} \frac{2}{9} & \frac{1}{3} & \frac{1}{9} \\ \frac{4}{9} & -\frac{1}{3} & \frac{2}{9} \\ \frac{1}{3} & 0 & -\frac{1}{3} \end{bmatrix}$$

Use $[A]^{-1}$ to solve the original matrix equation for the variable matrix, $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

$$[A]^{-1}[A]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [A]^{-1}[B]$$

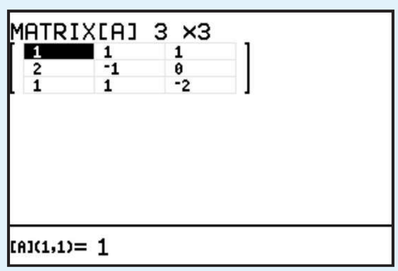
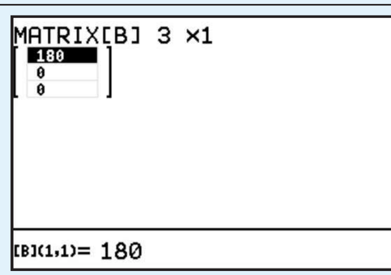
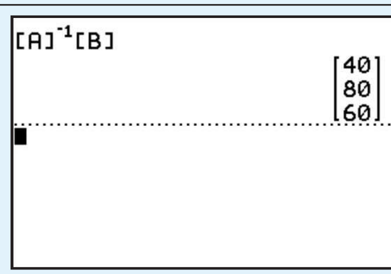
$$\begin{bmatrix} \frac{2}{9} & \frac{1}{3} & \frac{1}{9} \\ \frac{4}{9} & -\frac{1}{3} & \frac{2}{9} \\ \frac{1}{3} & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{9} & \frac{1}{3} & \frac{1}{9} \\ \frac{4}{9} & -\frac{1}{3} & \frac{2}{9} \\ \frac{1}{3} & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 180 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 40 \\ 80 \\ 60 \end{bmatrix}$$

The solution to the given system is (40, 80, 60). In the context of the problem, the measures of the three interior angles of ΔABC are 40°, 80°, and 60°.

USING TECHNOLOGY TO SOLVE SYSTEMS OF THREE LINEAR EQUATIONS WITH MATRICES

Graphing technology, such as a graphing calculator or app, can be extremely beneficial when solving systems of equations with matrices.

Write the system of linear equations as a matrix equation. Make sure that all three linear equations are in standard form.	$\begin{cases} x + y + z = 180 \\ y = 2x \\ x + y = 2z \end{cases} \rightarrow \begin{cases} x + y + z = 180 \\ 2x - y + 0z = 0 \\ x + y - 2z = 0 \end{cases}$ $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 180 \\ 0 \\ 0 \end{bmatrix}$
Enter the coefficient matrix into one matrix of your graphing technology.	
Enter the constant matrix into a second matrix of your graphing technology.	
Use matrix operations to calculate $[A]^{-1}[B]$.	

SYSTEMS OF THREE LINEAR EQUATIONS WITH MATRICES

Matrices can be used to represent and solve systems of three linear equations.

- Make sure that all three linear equations are in standard form, $Ax + By + Cz = D$, where A, B, C, and D are integers and at least one of A, B, and C is not equal to 0.
- Use the matrix equation $[A]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [B]$ where $[A]$ represents the coefficient matrix and $[B]$ represents the constant matrix.
- Determine the inverse of the coefficient matrix, $[A]^{-1}$.
- Left-multiply both sides of the matrix equation by $[A]^{-1}$. The left member of the equation, $[A]^{-1}[A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, is the identity matrix.
- Technology can be used to enter and calculate the values of the variable matrix, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [A]^{-1}[B]$.

EXAMPLE 1

Points X and Y are between points W and Z. The distance between points W and Z is thirty-six millimeters. The distance between points W and X is half the distance from point X to point Y. The distance between points Y and Z is one millimeter less than the distance from point X to point Y. What is the length in millimeters of each segment? Write a system of three linear equations to represent the situation and a matrix equation to represent the system. Represent and solve the system using matrices with technology.

STEP 1 **Define variables to represent the unknowns and use them write a system of linear equations to represent the system.**

Since the problem asks you to determine the length of the segments, the variables will represent the lengths, in millimeters, of the three segments. Let x represent the length of $\overline{WX}$, let y represent the length of $\overline{XY}$, and let z represent the length of $\overline{YZ}$.

"The segment shown is a total of thirty-six millimeters long." $\rightarrow x + y + z = 36$.
"The distance between points W and X is half the distance from point X to point Y." $\rightarrow x = \frac{1}{2}y$.
"The distance between points Y and Z is one millimeter less than the distance from point X to point Y." $\rightarrow z = y - 1$.

Therefore, a system of linear equations that represents the situation is

$$\begin{cases} x + y + z = 36 \\ x = \frac{1}{2}y \\ z = y - 1 \end{cases}$$

STEP 2 **Write a matrix equation that corresponds to the system of linear equations you wrote in Step 1. If necessary, rewrite any linear equations that are not already in standard form before writing the matrix equation.**

Two of the linear equations in the system you wrote in Step 1 are not in standard form, so it is necessary to rewrite both equations. Remember that if an equation does not contain all three variables, you should use a term with 0 as a coefficient as a placeholder.

$$\begin{cases} x + y + z = 36 \\ x = \frac{1}{2}y \\ z = y - 1 \end{cases} \rightarrow \begin{cases} x + y + z = 36 \\ x - \frac{1}{2}y + 0z = 0 \\ 0x - y + z = -1 \end{cases}$$

Although some of the variables appear to have no coefficients, remember that a coefficient of one is implied. A matrix equation that represents the system above is

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{\text{coefficient matrix}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\text{variable matrix}} = \underbrace{\begin{bmatrix} 36 \\ 0 \\ -1 \end{bmatrix}}_{\text{constant matrix}}.$$

STEP 3 **Represent and solve the system using matrices with technology.**

To represent the system using technology, enter a 3×3 matrix A with entries equal to the coefficient matrix and a 3×1 matrix B with entries equal to the constant matrix.

To solve the system using technology, multiply the inverse of matrix A by matrix B.

$$A^{-1}B = \begin{bmatrix} 7.4 \\ 14.8 \\ 13.8 \end{bmatrix}$$

$$\begin{cases} x + y + z = 36 \\ x = \frac{1}{2}y \\ z = y - 1 \end{cases}$$ *is the system of linear equations that represents this situation if x is the length in millimeters of $\overline{WX}$, y is the length in millimeters of $\overline{XY}$, and z is the length in millimeters of $\overline{YZ}$.*

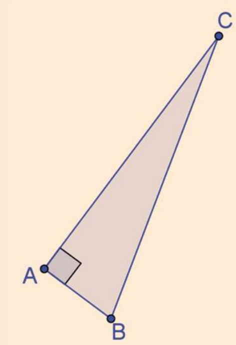
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 36 \\ 0 \\ -1 \end{bmatrix}$$ *is the matrix equation that represents the system of equations.*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7.4 \\ 14.8 \\ 13.8 \end{bmatrix}$$ *is the solution for the system determined using matrices and technology.*

The length of $\overline{WX}$ is 7.4 mm, the length of $\overline{XY}$ is 14.8 mm, and the length of $\overline{YZ}$ is 13.8 mm.

YOU TRY IT! #1

The perimeter of the right triangle shown is six hundred fifty centimeters. The length of its hypotenuse is one centimeter more than the length of its longer leg. The length of the longer leg of the right triangle is sixty-three centimeters less than fifteen times the length of its shorter leg. How long is each side of the right triangle? Write a system of three linear equations to represent the situation and a matrix equation to represent the system. Represent and solve the system using matrices with technology.



EXAMPLE 2

A baseball stadium has three levels of seats: field level, mezzanine level, and upper level. Field level seat tickets sell for $29 each. Mezzanine level seat tickets sell for $19.99 apiece. Upper level seat tickets sell for $13.50. There are as many field level seats as there are mezzanine and upper level seats combined. The stadium has 58,000 seats and takes in $1,336,340 for every sold out game. How many seats are in each level of the baseball stadium? Write a system of three linear equations to represent the situation and a matrix equation to represent the system. Represent and solve the system using matrices with technology.

STEP 1 **Define variables to represent the unknowns and use them write a system of linear equations to represent the system.**

Since the problem asks you to determine how many seats are in each level of the baseball stadium, the variables will represent the number of seats in each level. Let f represent the number of seats in the field level, let m represent the number of seats in the mezzanine level, and let u represent the number of seats in the upper level of the baseball stadium.

"There are as many field level seats as there are mezzanine and upper level seats combined." $\rightarrow f = m + u$.
"The stadium has 58,000 seats…." $\rightarrow f + m + u = 58{,}000$.
"Field level seat tickets sell for \$29 each. Mezzanine level seat tickets sell for \$19.99 apiece. Upper level seat ticket prices are \$13.50. …and takes in \$1,336,340 for every sold out game" $\rightarrow 29f + 19.99m + 13.50u = 1{,}336{,}340$.

Therefore, a system of linear equations that represents the situation is

$$\begin{cases} f = m + u \\ f + m + u = 58{,}000 \\ 29f + 19.99m + 13.50u = 1{,}336{,}340 \end{cases}$$

Step 2 **Write a matrix equation that corresponds to the system of linear equations you wrote in Step 1. If necessary, rewrite any linear equations that are not already in standard form before writing the matrix equation.**

One of the linear equations in the system you wrote in Step 1 is not in standard form, so it is necessary to rewrite that equation.

$$\begin{cases} f = m + u \\ f + m + u = 58{,}000 \\ 29f + 19.99m + 13.50u = 1{,}336{,}340 \end{cases} \rightarrow \begin{cases} f - m - u = 0 \\ f + m + u = 58{,}000 \\ 29f + 19.99m + 13.50u = 1{,}336{,}340 \end{cases}$$

Although some of the variables appear to have no coefficients, remember that a coefficient of one is implied. A matrix equation that represents the system above is

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 29 & 19.99 & 13.50 \end{bmatrix} \begin{bmatrix} f \\ m \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 58{,}000 \\ 1{,}336{,}340 \end{bmatrix}.$$

STEP 3 **Represent and solve the system using matrices with technology.**

To represent the system using technology, enter a 3×3 matrix A with entries equal to the coefficient matrix and a 3×1 matrix B with entries equal to the constant matrix.

To solve the system using technology, multiply the inverse of matrix A by matrix B.

$$A^{-1}B = \begin{bmatrix} 29{,}000 \\ 16{,}000 \\ 13{,}000 \end{bmatrix}$$

$\begin{cases} f = m + u \\ f + m + u = 58{,}000 \\ 29f + 19.99m + 13.50u = 1{,}336{,}340 \end{cases}$ ***is the system of linear equations that represents this situation where f is the number of seats in the field level, m is the number of seats in the mezzanine level, and u is the number of seats in the upper level of the baseball stadium.***

$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 29 & 19.99 & 13.50 \end{bmatrix}\begin{bmatrix} f \\ m \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 58{,}000 \\ 1{,}336{,}340 \end{bmatrix}$ ***is the matrix equation that represents the system of equations.***

$\begin{bmatrix} f \\ m \\ u \end{bmatrix} = \begin{bmatrix} 29{,}000 \\ 16{,}000 \\ 13{,}000 \end{bmatrix}$ ***is the solution for the system determined using matrices and technology.***

The baseball stadium has 29,000 seats in its field level, 16,000 seats in its mezzanine level, and 13,000 seats in its upper level.

YOU TRY IT! #2

A snack company sells bulk nuts, fruits, and granola. The company plans to offer a new trail mix for \$4.07 per pound that is composed of granola, peanuts, and raisins. There will be twice as many pounds of peanuts in the new trail mix as raisins. When sold separately, granola sells for \$5.99 per pound, peanuts sell for \$2.99 per pound, and raisins sell for \$3.99 per pound. How many pounds of each ingredient are in 100 pounds of the new trail mix? Write a system of three linear equations to represent the situation and a matrix equation to represent the system. Represent and solve the system using matrices with technology.

PRACTICE/HOMEWORK

For questions 1 – 3, create a matrix equation to represent each system of equations.

1. $\begin{cases} x + 2y - 3z = -2 \\ 2x - 2y + z = 7 \\ 2x + y + 3z = -4 \end{cases}$

2. $\begin{cases} 20a + 9b = 127 \\ 8a + 18b + 3c = 97 \\ 3b + 5c = 14 \end{cases}$

3. $\begin{cases} x + 2y + 3z = 9 \\ x = -2y \\ x + 4y - z = -5 \end{cases}$

Use the situation described below to answer questions 4 – 8.

FINANCE

Carla wants to order gift bags of mixed dried fruits. Three of her options are described in the table below.

DRIED FRUIT BAGS

	DESCRIPTION	COST
MIXED BAG A	3 pounds each of pineapple and strawberries 2 pounds of apples	\$43.00
MIXED BAG B	4 pounds each of pineapple and apples 1 pound of strawberries	\$38.00
MIXED BAG C	5 pounds of pineapples 3 pounds of apples 2 pounds of strawberries	\$46.50

4. Write a system of equations that you could use to represent this situation. Let the variables x, y, and z represent the cost per pound of pineapple, apples, and strawberries, respectively.

5. Use the coefficients of x, y, and z to write matrix A where each row represents one equation and each column represents one of the variables x, y, and z.

6. Use the constants from each equation to write matrix B where each row represents one equation.

7. Use the matrix $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to represent the variables and write a matrix equation relating matrix A, matrix B, and the variable matrix.

8. Solve for $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and explain what each value means for the situation.

Use the situation described below to answer questions 9 – 11.

BUSINESS

A distribution center is sending a shipment of 420 shoes worth a total of \$19,740 to a local store. The shipment contains three types of shoes: Shoe A has a value of \$50, Shoe B has a value of \$65, and Shoe C has a value of \$33.50. There are twice as many of shoe “C” than there are of shoe “B”.

9. Write a system of equations that you could use to represent this situation. Let the variables a, b, and c represent the number of each type of shoe (A, B, and C) respectively.

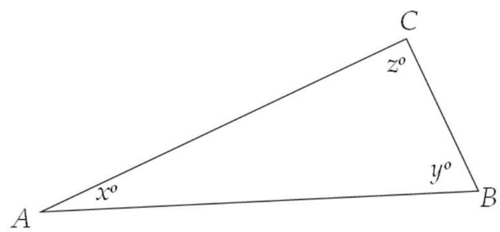
10. Create a matrix equation to represent this system of equations.

11. Solve for $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, and explain what each value means for the situation.

Use the situation described below to answer questions 12 – 14.

GEOMETRY

In the ΔABC, $m\angle C$ is three times $m\angle A$. Also, the sum of $m\angle A$ and $m\angle B$ is equal to $m\angle C$. Remember that in all triangles, the sum of the measures of the interior angles is 180°.



12. Write a system of equations that you could use to represent this situation.

13. Create a matrix equation to represent this system of equations.

14. Solve for $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and explain what each value means for the situation.

Use the situation described below to answer questions 15 – 17.

FINANCE

Raul has an uncle who likes to play games when he visits. On his most recent visit, he told Raul that he could have all the money in his wallet if he were able to guess how many of each kind of bill he had. Here are the clues he gave:

- I have only 3 denominations of bills - \$5 bills, \$10 bills, and \$20 bills.
- I have a total of 17 bills in my wallet.
- The value of all the money in my wallet is \$180.
- I have twice as many \$5 bills as I do \$10 bills.

15. Write a system of equations that you could use to represent this situation. Let the variables a, b, and c represent the number of each type of bill (\$5, \$10, and \$20) respectively.

16. Create a matrix equation to represent this system of equations.

17. Solve for $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, and explain what each value means for the situation.

Use the situation described below to answer questions 18 – 20.

FINANCE

A school's Theater Club is performing a play for the general public. The table below shows the cost of each type of ticket the Theater Club will sell.

TICKET PRICES	
CHILD (UP TO 11 YEARS)	$2.50
STUDENT (12 – 18 YEARS)	$5.00
ADULT (19 YEARS AND UP)	$7.50

They sold 600 tickets and took in $3,085. The number of Student tickets sold was twice the sum of Child and Adult tickets.

18. Write a system of equations that you could use to represent the number of each type of ticket. Let the variables c, s, and a represent the number of each type of ticket (child, student, adult) respectively. Rewrite the system so that all equations are in standard form.

19. Create a matrix equation to represent this system of equations.

20. Solve the matrix equation, and explain what each value means for the situation.

Chapter 6 Review

Use the data below to answer questions 1 – 3.

POPULATION OF 5 LARGEST CITIES IN TEXAS

CITY	1990	2000	2010
AUSTIN	465,622	656,562	790,390
DALLAS	1,006,877	1,188,580	1,197,816
FT. WORTH	447,619	534,694	741,206
HOUSTON	1,630,553	1,953,631	2,099,451
SAN ANTONIO	935,933	1,144,646	1,327,407

Source: Texas State Library

1. Create matrix M to represent this data.

2. What is the value of entry $m_{4,3}$ and what does it represent?

3. Which city had the greatest percentage increase between 1990 and 2010?

Use the data in the matrices below to answer questions 4 – 6.

Matrix R represents the revenue collected by Tina's T's T-shirt shop for three products over a period of 4 months. Matrix C represents the costs (including wholesale costs, advertising, and stocking fees) to the company for each of these products over the same time period.

$$R = \begin{array}{r} \\ \text{T-shirts} \\ \text{Caps} \\ \text{Purses} \end{array} \begin{array}{c} \begin{array}{cccc} \text{January} & \text{February} & \text{March} & \text{April} \end{array} \\ \begin{bmatrix} 360 & 408 & 576 & 720 \\ 125 & 175 & 275 & 350 \\ 270 & 162 & 216 & 324 \end{bmatrix} \end{array} \quad C = \begin{array}{r} \\ \text{T-shirts} \\ \text{Caps} \\ \text{Purses} \end{array} \begin{array}{c} \begin{array}{cccc} \text{January} & \text{February} & \text{March} & \text{April} \end{array} \\ \begin{bmatrix} 216 & 245 & 346 & 432 \\ 72.5 & 97.5 & 148 & 184 \\ 179 & 103 & 141 & 217 \end{bmatrix} \end{array}$$

4. Create matrix P to represent the profit on sales of these three items.

5. Which entry shows the greatest monthly profit?

Use the following scenario for questions 6 – 7.

For the products at Tina's T-shirt shop, she must pay extra fees for each dollar in revenue she collects. Matrix R below shows the revenue collected for T-shirts and caps during the months of January, February, March and April.

$$R = \begin{bmatrix} 360 & 408 & 576 & 720 \\ 125 & 175 & 275 & 350 \\ 270 & 162 & 216 & 324 \end{bmatrix}$$

6. If Tina must charge 7.5% sales tax on each item, create matrix T to represent the amount of the tax that will be collected. Round each value to the nearest penny.

7. To prepare for a new product line, Tina offers a 50% discount on all of her old products. How would this have affected her revenue if she had applied the discount to the old products over the four months represented in matrix R? Create matrix D with the discount before taxes.

Use the following matrices to answer questions 8 – 12.

$$A = \begin{bmatrix} 4 & -1 \\ 0 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -4 & 0 & 2 \\ 7 & 1 & -5 \end{bmatrix} \quad C = \begin{bmatrix} 9 \\ -2 \\ 6 \end{bmatrix} \quad D = \begin{bmatrix} 1 & -3 \\ 0 & 1 \\ 2 & 4 \end{bmatrix} \quad E = \begin{bmatrix} 5 & -4 & 2 \end{bmatrix}$$

8. Which pairs of matrices can be multiplied in either order?

Find each product:

9. AB **10.** DC **11.** EC

Use the following scenario for questions 12 – 15.

The U-Box-It store carries storage boxes in four sizes. The tables below show the four sizes of storage boxes that the store sells.

U-BOX-IT STORAGE BOX SALES

	SMALL	MEDIUM	LARGE	X-LARGE
JUNE	12	35	20	5
JULY	18	42	33	21
AUGUST	20	28	24	10

STORAGE BOX PRICES

	PRICE
SMALL	$4.50
MEDIUM	$5.25
LARGE	$7.00
X-LARGE	$9.50

12. Create Matrix B to show how many of each size box were sold during the summer months.

13. Create Matrix P to show the price of each size box.

14. Find matrix R, the product of matrices B and P, to show the revenue generated from selling storage boxes.

15. What is the meaning of entry $r_{2,1}$?

Create a matrix equation to represent each system of linear equations, then solve the system.

16. $\begin{cases} 4x + 5y = -7 \\ -2x - 3y = 5 \end{cases}$

17. $\begin{cases} 4x + 2 = 5y \\ 10y = 7 + 2x \end{cases}$

Use the scenario below to answer questions 18 – 19.

The band sold candy bars as a fundraiser. They sold a total of 450 candy bars for a total of \$961.50. Each chocolate bar was sold for \$2 while each peanut cluster was sold for \$2.50. Find the number of chocolate bars, c, and the number of peanut clusters, p, that were sold.

18. Write a system of equations to represent this scenario.

19. Write a matrix equation to represent the system and solve for c and p.

Create a matrix equation to represent each system of linear equations then solve the system.

20. $\begin{cases} 2x - 4y + z = 1 \\ 5x = -3z \\ x + y - z = 8 \end{cases}$

21. $\begin{cases} 2x - z = 0 \\ x + y = 4.6 \\ y = z + 1 \end{cases}$

Use the scenario below to answer questions 22 – 24.

Jason has a collection of nickels, dimes, and quarters that have a total value of \$2.95. Of the 24 coins, the number of nickels and quarters combined is two more than the number of dimes.

22. Write a system of equations to represent this scenario, using n, d, and q to represent the number of nickels, dimes, and quarters, respectively.

23. Create a matrix equation to represent the system.

24. How many of each type of coin does Jason have?

MULTIPLE CHOICE

25. Matrix K and matrix M are shown below. Which matrix represents the results of $[K] + [M]$?

$$K = \begin{bmatrix} -5 & 9 & -4 \\ 10 & 8 & 3 \\ -4 & 6 & 12 \end{bmatrix} \quad M = \begin{bmatrix} 0 & -3 & 5 \\ 7 & -1 & 8 \\ 4 & -6 & 10 \end{bmatrix}$$

A. $\begin{bmatrix} -5 & 12 & 1 \\ -3 & -8 & 11 \\ 8 & -12 & 22 \end{bmatrix}$ **B.** $\begin{bmatrix} -5 & 6 & 1 \\ 17 & 7 & 11 \\ 0 & 0 & 22 \end{bmatrix}$ **C.** $\begin{bmatrix} 0 & -27 & -20 \\ 70 & -8 & 24 \\ -16 & -36 & 120 \end{bmatrix}$ **D.** $\begin{bmatrix} 5 & -12 & 9 \\ -3 & -9 & 5 \\ 8 & -12 & -2 \end{bmatrix}$

26. Given matrices $R = \begin{bmatrix} 4 & -1 \\ 0 & 5 \end{bmatrix}$ and $T = \begin{bmatrix} -4 & 0 & 2 \\ 7 & 1 & -5 \end{bmatrix}$, what are the dimensions of the product RT?

A. 2 rows by 3 columns
B. 3 rows by 2 columns
C. 3 rows by 3 columns
D. 4 rows by 6 columns

27. Find the product MN for the matrices $M = \begin{bmatrix} 2 & 3 & -1 & 0 \\ 0 & 5 & 4 & -2 \end{bmatrix}$ and $N = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

A. Not possible **B.** $MN = \begin{bmatrix} 5 \\ 14 \end{bmatrix}$

C. $MN = \begin{bmatrix} 9 \\ 15 \end{bmatrix}$ **D.** $MN = \begin{bmatrix} 5 & 4 \\ 14 & 7 \end{bmatrix}$

28. What is the value of y in the solution to the system represented by the matrix equation below?

$$\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 17 \\ 69 \end{bmatrix}$$

A. 17
B. 69
C. 24
D. 31

29. Which matrix equation can be used to solve the system below?

$$\begin{cases} y = x + z \\ 3z = 5x + 2y \\ 4x - y + 2z = 8 \end{cases}$$

A. $\begin{bmatrix} 1 & 1 & -1 \\ 5 & 2 & 3 \\ 4 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 8 \end{bmatrix}$ **B.** $\begin{bmatrix} 1 & -1 & -1 \\ 3 & -5 & 2 \\ 4 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}$

C. $\begin{bmatrix} -1 & 1 & -1 \\ -5 & -2 & 3 \\ 4 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}$ **D.** $\begin{bmatrix} 1 & -1 & -1 \\ 3 & 5 & -2 \\ -1 & 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}$